27 August 2001

PHYSICS LETTERS A

Physics Letters A 287 (2001) 198–204

www.elsevier.com/locate/pla

Quasi-reversible instabilities of closed orbits

M.G. Clerc [b,c,*], P. Coullet [a], N. Vandenberghe [a], E. Tirapegui [c]

[a] *Institut Non-Linéaire de Nice, UMR 6618 CNRS-UNSA, 1361 Route des lucioles, F-06560 Valbonne, France*
[b] *Control and Dynamical Systems, 107-81, Caltech, Pasadena, CA 91125, USA*
[c] *Facultad de Ciencias Físicas y Mat., Depto. Física, Univ. de Chile, Casilla 487-3, Santiago, Chile*

Received 19 July 2000; received in revised form 28 June 2001; accepted 13 July 2001
Communicated by A.P. Fordy

Abstract

We characterize the three generic quasi-reversible instabilities of closed orbits: the quasi-reversible saddle-node, the Krein collision and the period doubling bifurcation. We show that after a periodic change of variables the asymptotic normal forms of the last two instabilities are the Maxwell–Bloch and the Lorenz equations. We exhibit a simple example of the quasi-reversible period doubling bifurcation, the quasi-reversible 2 : 1 resonance.

The study of bifurcations plays a central role in the modern theory of dynamical systems [1], and allows one to describe in a universal way phenomena which belong to different fields [2]. Recently in Ref. [3], we have characterized the instabilities of stationary solutions, which occur generically in one parameter families of finite-dimensional quasi-reversible dynamical systems. These are systems in which the terms which break the time reversal symmetry are small (irreversible effects), i.e., the system is in the neighborhood of a reversible one. The aim of this Letter is the characterization of the instabilities of periodic solutions of quasi-reversible systems. First, we shall describe the generic instabilities in one parameter families of periodic solutions for dissipative dynamical systems. After, we shall consider the instabilities of periodic solutions of reversible dynamical systems in the presence of small terms which break the time reversal symmetry (quasi-reversible systems). Finally, we shall study a simple example of the quasi-reversible period doubling instability, the quasi-reversible 2 : 1 resonance.

A closed orbit in phase space, that is a periodic solution, exhibits a bifurcation when the modulus of a Floquet multiplier associated to this orbit crosses the unit circle in the complex plane [4]. The local bifurcations which occur generically in one parameter families of finite-dimensional dissipative dynamical system are [4] (a) two complex conjugate Floquet multipliers cross the unit circle, (b) one real multiplier crosses the unit circle through $+1$, and (c) one real multiplier crosses the unit circle through -1.

The first bifurcation is characterized by the appearance of quasi-periodic motion [5]. The second is the saddle node bifurcation of a closed orbit, and the last one is the period doubling instability.

In reversible systems, i.e., systems which are invariant under a time reversal transformation (see the review [6] and references therein), the previous classification presents some changes. One has that if λ is a Floquet multiplier then λ^{-1} is also a multiplier

* Corresponding author.
E-mail address: marcel@dfi.uchile.cl (M.G. Clerc).

0375-9601/01/$ – see front matter
PII: S0375-9601(01)00490-X

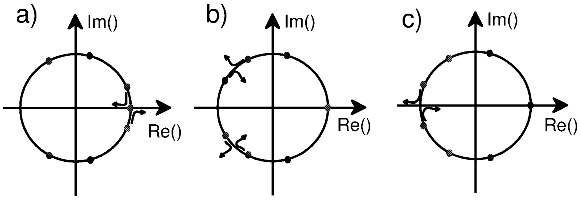

Fig. 1. Schematic representation of the Floquet multipliers for the generical instabilities of the reversible systems. (a) Reversible saddle node bifurcations, (b) Krein collision or or Neimark–Sacker bifurcation, and (c) the reversible period doubling.

for the reversed closed orbit [7,8]. Therefore the only "stable" or marginal states are those for which all Floquet multipliers are on the boundary of the unit circle (cf. Fig. 1), that is their modulus is equal to one. For this kind of systems the spectral instabilities in one parameter families are:

- The reversible saddle node bifurcation, i.e., two Floquet multipliers of modulus one on the unit circle have a collision in the point $+1$ of the real axis (see Fig. 1(a)).
- The Krein collision, which is the collision of two Floquet multipliers of modulus one and of their respective complex conjugates on the unit circle as it is illustrated in Fig. 1(b) [9,10]. This bifurcation is well known as the reversible Hopf bifurcation [8] or Neimark–Sacker bifurcation [11].
- The reversible period doubling bifurcation, i.e., two Floquet multipliers of modulus one which are on the unit circle become real through a collision at point -1 of the real axis as shown in Fig. 1(c).

A perturbation of the closed orbit in the direction of the orbit is marginal and has a Floquet multiplier equal to one. On the other hand, reversibility implies the transformation $\lambda \to \lambda^{-1}$ for Floquet multipliers, and consequently the multiplier one is degenerate. This is a consequence of the fact that, in reversible systems, one always has families of periodic solutions [7], and then a perturbation in the direction in which the family changes gives rise to another periodic solution with a period close to the period of the unperturbed solution.

The reversible saddle node bifurcation is described by four real variables $\{\theta, \rho, x, y\}$, where $\{\rho, x, y\}$ are the variables which describe the evolution in a neighborhood of the closed orbit [12]. The variables ρ and θ parametrize the family of periodic solutions and the closed orbit, respectively, and one has the time reversal transformation

$$t \to -t,\ \theta \to -\theta,\ \rho \to \pm\rho,\ x \to x,\ y \to -y,$$

where $\theta \to -\theta$ comes from the fact that the parametrization of the closed orbit changes its sign through time reversal. For the sake of simplicity henceforth we consider only the case $t \to -t$, $\theta \to -\theta$, $\rho \to \rho$, $x \to x$ and $y \to -y$. If the initial system is invariant under temporal translation ($t \to t + t_o$) then the phase variable θ does not couple to the other variables in the normal form, which is obtained after a T-periodic change of variables where T is the period of the closed orbit [4,13]. This property eliminates nonresonant periodic perturbations which play a fundamental role in the presence of structurally unstable objects such as homoclinic orbits. The asymptotic normal form is

$$\begin{aligned}
&\partial_t x = y,\\
&\partial_t y = \varepsilon \pm x^2 - \rho x + a\rho^2,\\
&\partial_t \rho = 0,\\
&\partial_t \theta = \frac{2\pi}{T} + f(\rho, x),
\end{aligned} \tag{1}$$

where f is a polynomial function, ε is the bifurcation parameter and a is a coefficient of order one. The scaling which leads to the previous equations is $\partial_{tt}x \sim O(\varepsilon)$, $\partial_t x \sim O(\varepsilon^{3/4})$, $\partial_t \rho \sim O(\varepsilon^{3/4})$, $x \sim O(\varepsilon^{1/2})$, $y \sim O(\varepsilon^{3/4})$, $\rho \sim O(\varepsilon^{1/2})$, and $\partial_t\theta \sim O(1)$. We add now the small irreversible effects which will appear as terms of the unfolding of the instability, and obtain asymptotically

$$\begin{aligned}
&\partial_{tt} x = \varepsilon \pm x^2 - \rho x + a\rho^2 - \nu \partial_t x,\\
&\partial_t \rho = \delta - \mu\rho + \eta x,\\
&\partial_t \varphi = c\rho + dx,
\end{aligned} \tag{2}$$

where $\varphi \equiv \theta - 2\pi t/T$, $\delta \sim O(\varepsilon^{3/4})$, $\mu \sim \nu \sim \eta \sim O(\varepsilon^{1/4})$, $d \sim c \sim 1$, and $\partial_t \varphi \sim O(\varepsilon^{1/2})$. The parameters μ and ν correspond to dissipative terms when they are positive, while δ and η are responsible for injection of energy. The dynamics around the closed orbit is given by the first two equations of the above set. These equations describe the appearance through a saddle node bifurcation of stable closed orbits, which can then lose stability through a Hopf–Andronov bifurcation. This corresponds to the appearance of quasi-periodic motion in the original system. When the stable limit cycle solution created in the Hopf–Andronov bifurcation intersects the unstable fixed point of the first two equations of (2) we have an homoclinic bifurcation and for a certain region of parameters the system exhibits Shilnikov chaos [14]. Hence, if the bifurcation parameter is increased, one peak of the Fourier spectrum moves towards zero and when it is close to zero, for certain values of the parameters, the system (2) shows chaotic dynamics which correspond to the period doubling route [15]. Note that, from the Fourier spectrum of a given signal, either experimental or numerical, one can identify the previous universal scenarios.

It is important to note that the energy is injected into the system through a forcing at the same frequency as the periodic solution. When the reversible system has more marginal modes (Floquet multipliers) without resonance between them, the quasi-reversible one is governed by the above equations, since the intensities of the other modes decreases in time.

In the case of the quasi-reversible Krein collision or Neimark–Sacker bifurcation, the system is described by six variables $\{A, B, \rho, \theta\}$ with A and B complex amplitudes. We consider the time reversal transformation

$$t \to -t,\ \theta \to -\theta,\ \rho \to \rho,\ A \to \bar{A},\ B \to \bar{B}.$$

Near the threshold of the bifurcation and after making a T-periodic change of variables and including the small irreversible terms the system is described by the asymptotic normal form

$$\begin{aligned}
\partial_t A &= i\Omega A + B,\\
\partial_t B &= -\big(\nu - i(\Omega + \Delta)\big)B + \varepsilon A \pm |A|^2 A - \rho A,\\
\partial_t \rho &= -\mu\rho + \eta |A|^2,\\
\partial_t \theta &= \frac{2\pi}{T} + f\big(\rho, |A|^2\big),
\end{aligned} \tag{3}$$

where f is a polynomial function, $\exp(i\Omega)$ the point of the unit circle where the two Floquet multipliers collide, ε the bifurcation parameter, Δ the detuning parameter, and the terms with coefficients ν, η, μ are irreversible terms. The scalings which lead to the previous equations are $\partial_t A \sim O(\varepsilon)$, $\partial_t B \sim O(\varepsilon^{3/2})$, $\rho \sim O(\varepsilon)$, $A \sim O(\varepsilon^{1/2})$, $B \sim O(\varepsilon)$, $\nu \sim \eta \sim \mu \sim \Delta \sim O(\varepsilon^{1/2})$ and $\partial_t\ \theta \sim O(1)$ and we can see that one has again the phase invariance $\theta \to \theta + \theta_o$ which decouples the first five equations from θ. These equations describe the $1:1$ resonance in presence of a neutral mode and in the absence of resonance, i.e., $\Omega T/2\pi \neq p/q$ with p and q integers. They are formally equivalent to the Maxwell–Bloch equations [16], which describe the interaction between two level atoms and an electromagnetic field [17].

The quasi-reversible period doubling is described, as the quasi-reversible saddle node bifurcation, with four variables $\{\theta, \rho, x, y\}$ which are obtained from the variables of the original problem through a $2T$-periodic change of variables [13]. But now as a consequence of the Floquet multiplier at -1, one has reflection invariance in the variables $\{x, y\}$, i.e., the normal form that describes the instability is invariant under the transformation $(x, y) \to (-x, -y)$. It is pertinent to recall here that for a stationary solution the saddle node and Hopf instabilities are the two local bifurcations which occur generically in one parameter families of finite-dimensional dissipative systems, and that the saddle node bifurcation, with the extra property of reflection symmetry, is the analogous of the period doubling instability of periodic solutions. The time reversal symmetry that we consider is

$$t \to -t,\ \theta \to -\theta,\ \rho \to \rho,\ x \to x, y \to -y,$$

and after taking into account the small quasi-reversible terms we obtain the following asymptotic normal form which describes the quasi-reversible system near the threshold of the instability:

$$\begin{aligned}
\partial_t x &= y,\\
\partial_t y &= \varepsilon x - x^3 - \rho x - \nu y,\\
\partial_t \rho &= -\mu\rho + \eta x^2,\\
\partial_t \theta &= \frac{2\pi}{T} + f\big(\rho, x^2\big).
\end{aligned} \tag{4}$$

Here f is a polynomial function, ε is the bifurcation parameter, the terms with coefficients ν and μ are dissipative when these coefficients are positive, henceforth we assume that they are positive, and the term proportional to η is a nonlinear injection or dissipation of energy. In order to obtain the latter equations we have considered the scalings $\partial_t x \sim O(\varepsilon)$, $\partial_t y \sim O(\varepsilon^{3/2})$, $\partial_t \rho \sim O(\varepsilon^{3/2})$, $x \sim O(\varepsilon^{1/2})$, $\rho \sim O(\varepsilon)$, $y \sim O(\varepsilon)$, $\nu \sim \eta \sim \mu \sim O(\varepsilon^{1/2})$ and $\partial_t \theta \sim O(1)$. Introducing the variable φ defined by $\theta = 2\pi t/T + \varphi$ the asymptotic normal form takes the form

$$\partial_{tt} x = \varepsilon x - x^3 - \rho x - \nu \partial_t x,$$

$$\partial_t \rho = -\mu\rho + \eta x^2,$$
$$\partial_t \varphi = m\rho + l x^2, \tag{5}$$

where $\partial_t \varphi$ is of order ε and we have only kept the dominant terms in the equations above. The first two equations are equivalent to the Lorenz model [3,18].

To illustrate the onset of this Lorenz bifurcation in the quasi-reversible period doubling of a closed orbit, we shall study as an example the truncated reversible 2 : 1 resonance [19], that is, we consider the interaction of the fundamental and the second harmonic mode, which can be written in terms of two complex amplitudes $\mathcal{A}$ and $\mathcal{B}$ which satisfy the equations

$$i\partial_t \mathcal{A} = i\Omega\mathcal{A} - a\mathcal{B}\bar{\mathcal{A}} + |\mathcal{A}|^2\mathcal{A} + c|\mathcal{B}|^2\mathcal{A},$$
$$i\partial_t \mathcal{B} = i2\Omega\mathcal{B} - b\mathcal{A}^2 - |\mathcal{B}|^2\mathcal{B} + d|\mathcal{A}|^2\mathcal{B}, \tag{6}$$

where $\bar{\mathcal{A}}$ is the complex conjugate of $\mathcal{A}$. Henceforth a is considered positive. Close to the stationary solution $\mathcal{A} = \mathcal{B} = 0$, $\mathcal{A}$ describes the mode of frequency Ω and $\mathcal{B}$ describes the mode of double frequency 2Ω. Introducing the variables $A = \mathcal{A}e^{i\Omega t}$ and $B = \mathcal{B}e^{i2\Omega t}$, the preceding equations read

$$i\partial_t A = -aB\bar{A} + |A|^2 A + c|B|^2 A,$$
$$i\partial_t B = -bA^2 - |B|^2 B + d|A|^2 B, \tag{7}$$

and are invariant under the time reversal transformation $t \to -t$, $A \to \bar{A}$, and $B \to \bar{B}$. The system presents a set of closed orbits of the form

$$A = 0, \qquad B = R_o e^{iR_o^2 t}, \tag{8}$$

where R_o parametrizes the family of closed orbits ($R_o \geqslant 0$). In order to study the stability of these periodic solutions, one can use the following ansatz:

$$B = (R_o + r)e^{i(R_o^2 t + \varphi)},$$
$$A = (\chi + iy)e^{i(R_o^2 t + \varphi)/2}, \tag{9}$$

which is a $2T$-periodic change of variables ($T = 2\pi/R_o^2$) from (A, B) to (r, φ, χ, y), where r is a perturbation of the amplitude of the solution, $R_o^2 t + \varphi$ parametrizes the closed orbit, and $\{\chi, y\}$ describe the evolution in the other transverse directions. Introducing the previous ansatz in Eqs. (7) and linearizing we obtain

$$\partial_t r = 0,$$

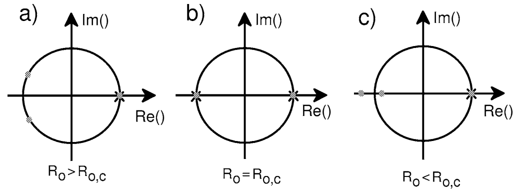

Fig. 2. Schematic representation of the Floquet multipliers of quasi-reversible 2 : 1 resonance (Eqs. (6)).

$$\partial_t \varphi = 2R_o \rho,$$
$$\partial_t y = -\left(\frac{R_o}{2} + cR_o - a\right)R_o \chi,$$
$$\partial_t \chi = \left(\frac{R_o}{2} + cR_o + a\right)R_o y.$$

The system always has a Floquet multiplier in one with multiplicity two which corresponds to the Jordan block of the two first equations. Let us define

$$R_{o,c} \equiv \frac{2a}{(1+2c)},$$

henceforth we assume that this quantity is positive, that is $c > -0.5$. For $R_o > R_{o,c}$ the system will exhibit two other complex conjugate multipliers of modulus one on the unit circle (cf. Fig. 2(a)), which become real after colliding at -1 on the real axis when $R_o = R_{o,c}$ (cf. Fig. 2(b)). For $R_o < R_{o,c}$ the two real multipliers start moving away from -1 in the real axis (cf. Fig. 2(c)). Hence, the periodic solution shows a period doubling bifurcation for $R_o = R_{o,c}$. Using Ansatz (9), the system around the bifurcation is described asymptotically by

$$\partial_t \rho = 0,$$
$$\partial_{tt} x = \varepsilon x - x\rho - x^3,$$
$$\partial_t \varphi = m\rho + l x^2,$$

where

$$\rho \equiv \frac{4a^3}{(1+2c)}\left(r + \frac{b(1+2c)}{4a^2}\chi^2\right),$$
$$x \equiv \frac{\chi}{s}, \qquad y = \frac{1+2c}{4a^2}s\partial_t x = \frac{1+2c}{4a^2}\partial_t \chi,$$
$$s \equiv \sqrt{\frac{4a^2}{1+2c}\left(1 + \frac{b}{4a}(1+2c) - d\right)},$$
$$\varepsilon \equiv \frac{8a^3}{(1+2c)^2}\left(a - \frac{R_o}{2} - cR_o\right), \qquad l = \frac{1}{a^2},$$

$$m=\left(\frac{b}{2a}(2c-1)-d\right)\Big/s^2. \tag{10}$$

Here ε is the bifurcation parameter ($\varepsilon \ll 1$) and $\{s, m, l\}$ are parameters of order one. When ε is positive the T-periodic closed orbit loses stability and gives rise to the appearance of a $2T$-periodic closed orbit. Let us consider the Poincaré section which cuts the closed orbit. At the onset of the bifurcation the characteristic time scale of the evolution of the variables $\{x, y, \rho\}$ is $\tau \sim O(\varepsilon^{-1/2}) \gg T$ and the Poincaré section will then show the evolution of the reduced dynamical system $\{x, y, \rho\}$.

We consider now the effect of the small terms which break the time reversal symmetry (quasi-reversible system). Eqs. (7) take the form

$$\begin{aligned} i\partial_t A &= -aB\bar{A}+|A|^2A+c|B|^2A \\ &\quad -i\gamma A+ie|A|^2A-ih|B|^2A, \\ i\partial_t B &= -bA^2-|B|^2B+d|A|^2B \\ &\quad +i\delta\left(R_o^2-|B|^2\right)B+iJ|A|^2B, \end{aligned} \tag{11}$$

where $\gamma, \delta, e, h, J \ll 1$. The terms proportional to γ and h are dissipative when these coefficients are positive, the linear term proportional to δR_o^2 is the injection of energy to the system and the nonlinear term proportional to δ is dissipative. Due to the irreversible terms, i.e., the ones proportional to γ, δ, e, h, and J, the family of closed orbits disappears and only one closed orbit persists, which has the form of (8), but now R_o is fixed. When one considers R_o close to $R_{o,c}$ ($|R_o - R_{o,c}| \ll 1$), the system exhibits a period doubling bifurcation. In contrast to the reversible case, the Floquet multiplier related to the variable r is now inside the unit circle, but near its boundary.

In order to study the dynamics around this bifurcation we make a $2T$-periodic change of variables (9) after which the system is described asymptotically by Eqs. (5), where x, ρ, φ, ε, m, and l are defined in (10), and

$$\nu = 2\left(\gamma + hR_{o,c}^2\right), \qquad \mu = \frac{8a^2\delta}{1+2c},$$
$$\eta = \frac{8a^3}{(1+2c)s^2}\left(\delta b + \frac{Ja}{1+2c}\right) - \frac{2b}{s^2}\left(\gamma + hR_{o,c}^2\right).$$

Note that the term proportional to e in Eqs. (11) does not play any role around the bifurcation. One could expect this result because this term gives rise to a cubic dissipative term. Since the irreversible terms do not break the phase invariance of the amplitudes A and B ($A \to Ae^{i\theta_o}$, $B \to Be^{i2\theta_o}$), the variable φ, which describes the phase, is not coupled to the other equations and does not appear in the first two equations. These are equivalent to the Lorenz model [20,21]

$$\begin{aligned} \dot{x}' &= \sigma(y' - x'), \\ \dot{y}' &= rx' \mp y' - x'z', \\ \dot{z}' &= -bz' + x'y', \end{aligned} \tag{12}$$

through the change of variables

$$\rho = z'\frac{(\eta+\mu)}{\tau_o} - \frac{x'^2}{\tau_o^2}, \qquad x = \frac{x'}{\tau_o},$$
$$\partial_t x = (y' - x')\frac{(\eta+\mu)}{\tau_o},$$

where $\tau_o = |(\eta+\mu)/(\nu-(\eta+\mu))|$, $\sigma = \eta+\mu$, $r = \varepsilon - (\eta+\mu)^2 + \nu(\eta+\mu)$, $b = \mu/\tau_o$, and "$\mp$" is the sign of the expression $-\nu + (\eta+\mu)$. When $-\nu + (\eta+\mu)$ is negative and σ positive the system can exhibit a stable homoclinic solution and in opposite case the homoclinic solution is unstable (σ positive). Therefore the system will show different routes to chaos. For the sake of simplicity, henceforth σ is assumed positive.

The classical scenario of Lorenz corresponds to $-\nu + (\eta+\mu)$ negative. The solution $x' = y' = z' = 0$ represents the periodic solution and goes through a pitchfork bifurcation which corresponds to the period doubling of the closed orbit. For a particular region of parameters, when the bifurcation parameter increases the system exhibits an unstable homoclinic bifurcation. The disappearance or explosion of this solution gives rise to two unstable periodic solutions [20] and to a chaotic saddle responsible for a chaotic transient [22], this type of behavior is usually called metastable chaos [23]. Then, a strong perturbation of the $2T$-periodic solution can display a chaotic transient. When one continues increasing the bifurcation parameter the unstable chaotic set changes its stability giving rise to the appearance of a stable strange attractor. The latter scenario is known as a *crisis* [24]. Thus, in this region of parameters the system will show as stable solutions the closed orbit of period $2T$ and a strange attractor around this periodic solution characterized by a Poincaré section which shows the Lorenz strange attractor. The unstable periodic solutions, created by the disappearance of the homoclinic pair, disappears when the control parameter is increased through a subcritical Hopf–Andronov bifurcation, with the stationary solutions created in the pitchfork bifurcation. Beyond this bifurcation the only stable solution exhibited by the system is chaotic [20].

When $-\nu + (\eta + \mu)$ is positive model (12) differs from the Lorenz model in a sign [21]. We have a different route to chaos, in which the solution $x' = y' = z' = 0$ still undergoes a pitchfork bifurcation (period doubling), the bifurcated solutions can exhibit a supercritical Hopf–Andronov bifurcation, and the $2T$-periodic solution loses stability and gives rise to a quasi-periodic motion (torus) with frequencies π/T and

$$\omega = \frac{2\mu\varepsilon}{\mu + \eta} + \nu\mu.$$

Increasing the bifurcation parameter the two symmetric periodic solutions of Eqs. (5) meet and give rise to a pair of homoclinic solutions (homoclinic bifurcation). Increasing further the bifurcation parameter, the system will display chaotic behavior through a cascade of homoclinic bifurcations, through a scenario known as a *gluing* [25]. Hence, when the bifurcation parameter is increased one frequency of the quasi-periodic solution vanishes (homoclinic bifurcation, gluing). Subsequently, the system will show a new quasi-periodic solution with a Poincaré section which is more complex than the previous one. Then, in the original system, if one increases the bifurcation parameter, the torus becomes unstable and gives rise to two symmetrical tori which after further increase of the parameter become an asymmetrical tori, i.e., the Poincaré section displays a pair of asymmetric periodic solutions. If one still increases the parameter these tori meet again, that is the Poincaré section exhibits an homoclinic bifurcation (gluing), and the explosion of this homoclinic solution pair gives rise to a more complex quasi-periodic motion, and it continues in the same way (cascade of gluings). As a result of this cascade the system exhibits a chaotic behavior which is characterized by a Lorenz strange attractor on the Poincaré section of the initial periodic solution (see Fig. 3).

In Refs. [26,27] a model of four variables has been considered which is a two mode truncation of the complex Ginzburg–Landau equation. This model has a quasi-periodic solution which disappears giving rise to chaotic behavior. Using a Poincaré section the author [27] has identified the topology of Lorenz and constructing the mapping of the consecutive maxima of one variable finds the characteristic Lorenz mapping [21]. This provides evidence that the motion is in a strange attractor, but it is important to remark that in the region of the parameters where the chaotic motion has been observed the model is not quasi-reversible. Summarizing, we have shown that at the onset of the quasi-reversible period doubling the normal form of the system is the Lorenz model, but the characteristic behavior with its topology can well persist far from the quasi-reversible region as illustrated in the model of references [26,27].

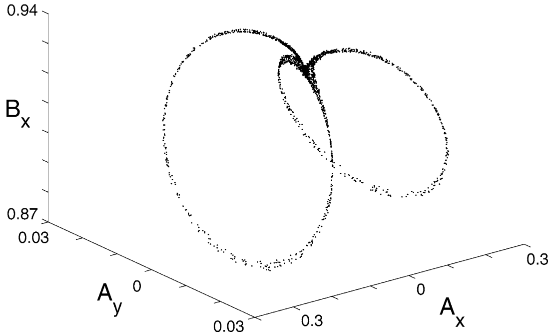

Fig. 3. Poincaré section at the onset of chaotic behavior obtained from numerical simulation of the quasi-reversal 2 : 1 resonance by the parameters $a = 0.5$, $b = 1$, $c = 0$, $\mu = 0.001$, $e = 0$, $h = 0$, $d = 0$, $\delta = 0.0446$, $J = 0$, and $R_o = 0.92$. The Poincaré section is defined by $B_y = 0$ and $\partial_t B_y > 0$.

The dynamical behavior of the 2 : 1 resonance has been extensively studied (see, for example, [19]). The quasi-reversible limit of this resonance allows us to characterize its dynamical behavior and in particular the chaotic one. Hence, the consideration of the quasi-reversible limit is a good strategy which permits to characterize a complex dynamical behavior of a system under study.

In general, it is very difficult to obtain the set of Eqs. (2)–(4) since one needs to know the periodic change of variables and consequently to solve explicitly a linear problem with periodic coefficients [4,13]. The example presented here is one of the few which can be described analytically. It is important to remark that the Floquet multipliers, the numerical simulations and the Poincaré sections, are the basic implements to describe the dynamic behavior near the closed orbit.

In conclusion, we have characterized the three generic codimension-one quasi-reversible instabilities of closed orbits: the quasi-reversible saddle-node, the Krein collision or Neimark–Sacker bifurcation and the period doubling bifurcations. The asymptotic normal forms of the last two bifurcations, obtained after a periodic change of variables, are equivalent to the Maxwell–Bloch equations and the Lorenz equations. Moreover, we have described the different dynamical scenarios which occur in the quasi-reversible period doubling bifurcation and illustrated the results with a simple example: the quasi-reversible 2 : 1 resonance.

Acknowledgements

The authors acknowledge support from the CNRS-CONICYT program, Fondecyt International Cooperation project, ECOS, and Cátedra Presidencial. One of us (M.C.) thanks the support of the Centro de Fisica No Lineal y Sistemas Complejos de Santiago.

References

[1] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer, New York, 1983.
[2] M. Cross, P. Hohenberg, Rev. Mod. Phys. 65 (1993) 851.
[3] M. Clerc, P. Coullet, E. Tirapegui, Phys. Rev. Lett. 83 (1999) 3820.
[4] V. Arnold, Chapitres Supplémentaires de la Thé orie des Equations Différentielles Ordinaires, MIR, Moscow, 1984.
[5] E. Ott, Chaos in Dynamical Systems, Cambridge University Press, New York, 1993.
[6] J.S.W. Lamb, Physica D 112 (1988) 1.
[7] R. Devaney, Ind. Univ. Math. J. 26 (1977) 247.
[8] M.B. Sevryuk, Reversible Systems, Lectures Notes in Math., Vol. 1211, Springer, Berlin, 1986.
[9] M.B. Sevryuk, A. Lahiri, Phys. Lett. A 154 (1991) 104.
[10] A. Bhowal, T.K. Roy, A. Lahiri, Phys. Rev. E 47 (1993) 3932.
[11] A. Lahiri, A. Bhowal, T.K. Roy, M.B. Sevryuk, Physica D 63 (1993) 99.
[12] J.K. Hale, Ordinary Differential Equations, Wiley, New York, 1980.
[13] G. Iooss, M. Adelmeyer, Topics in Bifurcations Theory, World Scientific, Singapore, 1999.
[14] M. Clerc, P. Coullet, E. Tirapegui, Int. J. Bifurc. Chaos 11 (2001) 591.
[15] A. Arnèodo, P.H. Coullet, E.A. Spiegel, C. Tresser, Physica D 14 (1985) 327.
[16] M. Clerc, P. Coullet, E. Tirapegui, Opt. Commun. 167 (1999) 159.
[17] A. Newell, J. Moloney, Nonlinear Optics, Addison–Wesley, Redwood, 1992.
[18] C.J. Marzec, E.A. Spiegel, SIAM J. Appl. Math. 38 (1980) 403.
[19] E. Knobloch, M.R.E. Proctor, Proc. R. Soc. London A 415 (1988) 61.
[20] C. Sparrow, The Lorenz Equations: Bifurcation, Chaos, and Strange Attractors, Springer, New York, 1982.
[21] E. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[22] H.E. Nusse, J.A. Yorke, Physica D 36 (1989) 137.
[23] J.A. Yorke, E.D. Yorke, J. Stat. Phys. 21 (1979) 263.
[24] C. Grebogi, E. Ott, J.A. Yorke, Phys. Rev. Lett. 48 (1982) 1507.
[25] Arnèodo, P. Coullet, C. Tresser, Phys. Lett. A 81 (1981) 197.
[26] A. Doelman, Nonlinearity 4 (1991) 231.
[27] H. Moon, Phys. Rev. Lett. 79 (1997) 403.