

How vortices mix

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The advection of a passive scalar blob in the deformation field of an axisymmetric vortex is a simple mixing protocol for which the advection–diffusion problem is amenable to a near-exact description. The blob rolls up in a spiral which ultimately fades away in the diluting medium. The complete transient concentration field in the spiral is accessible from the Fourier equations in a properly chosen frame. The concentration histogram of the scalar wrapped in the spiral presents unexpected singular transient features and its long time properties are discussed in connection with real mixtures.

1. Introduction

A central question in scalar mixing is the satisfactory description of the histogram or probability density function (PDF) $P(c)$ of the concentration levels c of a substance being mixed. The question is particularly interesting, and relevant to many applications, when the substrate is stirred since in that case molecular diffusion is altered, and in most cases enhanced, by the underlying substrate motions.

The interplay between molecular diffusion and simple deformation fields is a classical problem. It has been solved in a closed form in a variety of situations such as saddle point flow, simple shear in two dimensions (Ranz 1979; Moffatt 1983) and in three dimensions (Villermaux & Rehab 2000), and in axisymmetric point vortex (Rhines & Young 1983; Flohr & Vassilicos 1997) or spreading vortex flow (Marble 1988; Bajer, Bassom & Gilbert 2001).

Most attention has focused on the kinetics of the diffusion process in the presence of stirring motion, particularly its dependence on the substrate rate of deformation γ , and diffusion properties of the scalar (diffusivity D). Regarding the characteristic time t_s after which fluctuations start to decay from an initial scalar spatial distribution, of crucial importance is the rate at which material lines grow in time due to the substrate motions (Villermaux 2002). If material lines grow as γt , as is the case in a point vortex flow, the mixing time of, say, a scalar blob of initial size s_0 is $t_s \sim \gamma^{-1} Pe^{1/3}$; if material surfaces in three dimensions grow as $(\gamma t)^2$, then $t_s \sim \gamma^{-1} Pe^{1/5}$ and if material lines are exponentially stretched as $e^{\gamma t}$, then $t_s \sim (2\gamma)^{-1} \log Pe$ where $Pe = \gamma s_0^2/D$ is a Péclet number.

The times t_s given above are the relevant mixing times once the inverse of the elongation rate γ^{-1} is smaller than the diffusive time of the blob based on its initial size s_0^2/D , that is for $Pe > 1$. In the limit $Pe \gg 1$, t_s is essentially given by the time needed to deform the blob γ^{-1} , and molecular diffusion, although a crucial step in the ultimate mixing, plays only a weak correction role in the kinetics of the process.

Experiments or numerical simulations addressing this problem quantitatively are scarce, and are mostly limited to short times (i.e. $t \lesssim t_s$), therefore reflecting more the

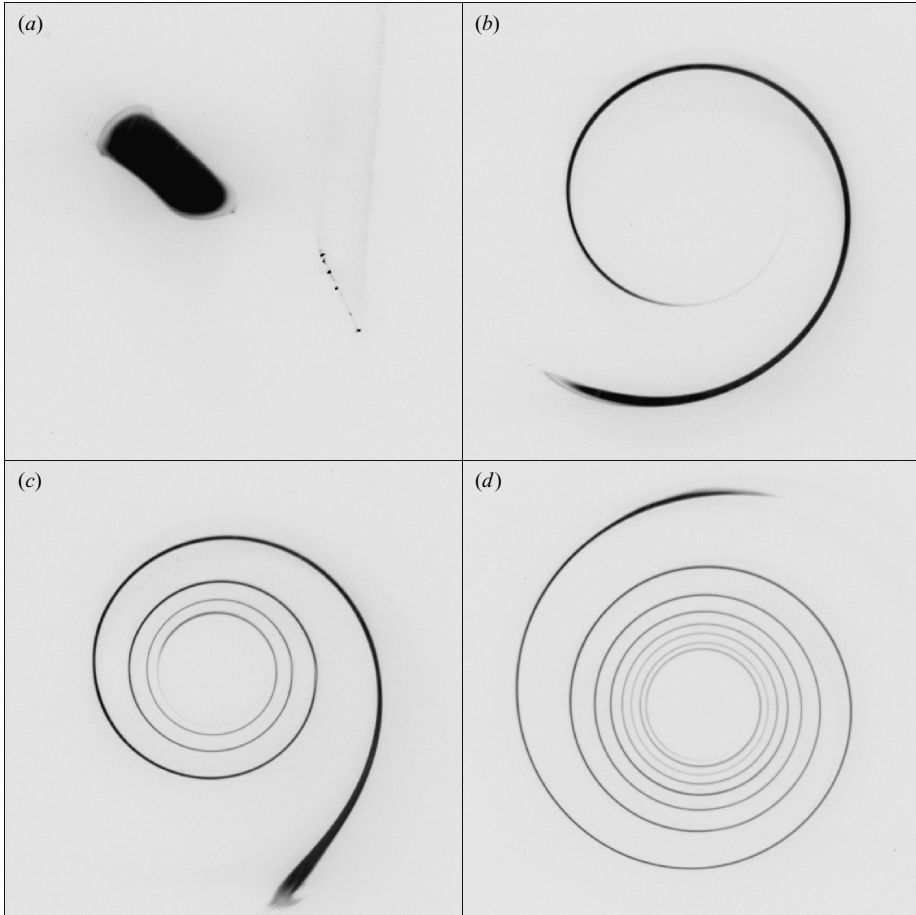


FIGURE 1. Roll-up of a blob of fluorescent dye in a point vortex at (a) $t = 0$, (b) $t = 2$ s, (c) $t = 5$ s and (d) $t = 10$ s. Each picture covers a field 4.8×4.8 cm² wide and the circulation of the vortex is 14.2 cm² s⁻¹. The data come from experiments described in § 2.

kinematics of the flow than its mixing properties (see, however Cetegen & Mohamad 1993 and Verzicco & Orlandi 1995).

Based on a spatially and temporally resolved experiment, we study the mixing chronology of a blob of dye embedded in the displacement field of a diffusing, Lamb–Oseen type vortex. The process is described, from the initial segregation of the blob to a state where it is almost completely diluted in the surrounding medium, through the evolution of the spatial scalar field, and associated transient evolution of the overall concentration distribution $P(c)$.

2. A diffusive spiral

2.1. Chronology

The phenomenon we analyse is illustrated on figure 1. A uniform blob of dye (the dark patch shown on figure 1a) is deposited in a still transparent medium. Then a vortex is formed by the roll-up of a vortex sheet in the vicinity of the blob, which wraps around the vortex as seen on figure 1(b). Although it now has a thin transverse size, most of

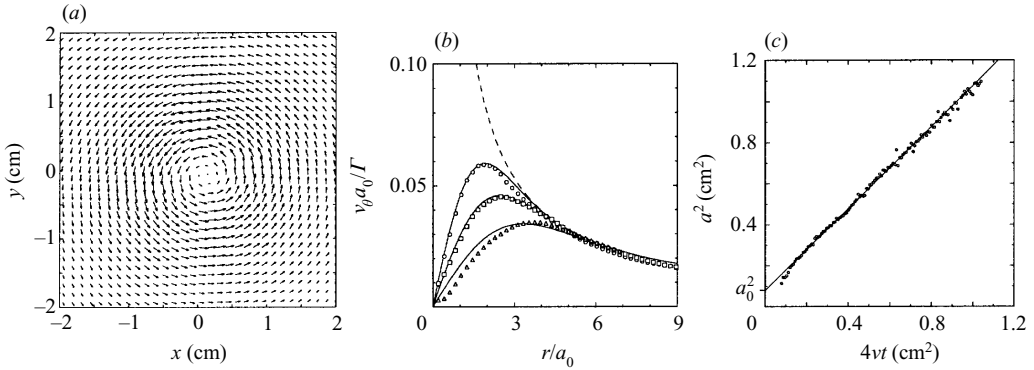


FIGURE 2. (a) Velocity field in the plane of the vortex at $t = 10$ s. (b) Radial profiles of the azimuthal velocity measured at $t = 5$ s (\circ), $t = 10$ s (\square) and $t = 20$ s (\triangle). Solid lines correspond to the profiles expected from a Lamb–Oseen vortex defined by (2.1) with $\Gamma = 14.2 \text{ cm}^2 \text{ s}^{-1}$ and $a_0 = 0.3$ cm. The dashed line corresponds to a point vortex defined by (3.1). (c) Core size of the Lamb–Oseen vortex measured by a least-square fit of the two-dimensional measured velocity field and compared to (2.2) (solid line).

the fluid particles constituting the blob still have the initial concentration. The blob deforms in a spiral shape and after four turns (figure 1c), the dye concentration is no longer uniform along the spiral: it is weaker near the centre of the vortex where the spiral is very thin, and still close to the injection concentration in the outer region of the spiral which is thicker. On figure 1(d), the spiral has made more than seven turns and is about to vanish in the diluting medium. The thickness of the spiral is fairly constant.

Molecular diffusion has clearly been enhanced by the vortex motion. The time lapse between figures 1(a) and 1(d) is 10 s, whereas the time scale of pure diffusion based on the initial size s_0 of the blob s_0^2/D is about 10^3 s.

2.2. Flow field

The vortex is formed by the impulsive flapping motion of a long flat plate in a large tank of water initially at rest. The vorticity layer formed on the surface of the plate rolls up and detaches from the plate end, producing an axisymmetric vortex which remains two-dimensional long after the dye has been mixed. A thin uniform argon-ion laser sheet is shone through the tank perpendicular to the plate, and the two-dimensional motion of the vortex is analysed by particle image velocimetry (PIV) using a Kodak 1008×1018 pixels digital camera aimed perpendicular to the laser sheet. Further information on the set-up and PIV techniques can be found in Meunier & Leweke (2002a) and Meunier & Leweke (2002b) respectively.

The dye is introduced, prior to the formation of the vortex, by a small tube positioned below the laser sheet, and forming a slowly ascending column of dye, aligned with the vortex axis. The dye concentration field (disodium fluoresceine with initial concentration $c_0 \approx 10^{-3} \text{ mol l}^{-1}$) is recorded with the same camera and stored on a disk. The overall framing rate allows a complete roll-up sequence to be temporally resolved. The images are digitized on 8 bits and the resulting background subtracted grey levels are proportional to the dye concentration.

Figure 2(a) shows an example of the axisymmetric velocity field obtained by PIV after the vortex creation. The radial profiles of azimuthal velocity v_θ shown on figure 2(b) agree well with that of a Lamb–Oseen vortex, defined in the cylindrical

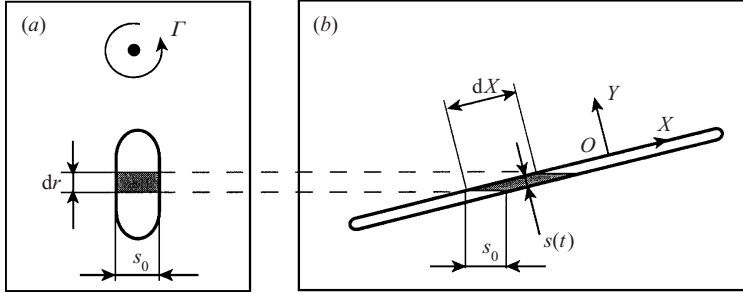


FIGURE 3. Schematic of the scalar blob elongation: (a) initial state and (b) at time t .

coordinates (r, θ, z) by

$$v_\theta = \frac{\Gamma}{2\pi r} (1 - e^{-r^2/a^2}) \quad (2.1)$$

Here, $\Gamma = 14.2 \text{ cm}^2 \text{ s}^{-1}$ is the circulation of the vortex, and a its core size. This vortex is an exact solution of the Navier–Stokes equations provided that

$$a^2 = a_0^2 + 4\nu t, \quad (2.2)$$

where ν is the kinematic viscosity of the fluid, which is in close agreement with the observed growth (figure 2c), a_0 being the initial vortex radius equal to 0.3 cm.

The dashed line in figure 2(b) is the velocity profile of a point vortex with the same circulation, defined by (3.1) below. It is tangent to the measured velocity profiles for large radii ($r/a_0 > 3$).

To decouple the problem of mixing from the (trivial) problem of the temporal evolution of the velocity field itself, we have systematically deposited the blob of dye far enough from the vortex core so that the velocity field remains that of a steady point vortex throughout the whole mixing process.

3. Concentration field along the spiral

We consider the evolution of a blob of dye of initial size s_0 , in the two-dimensional, incompressible flow of a point vortex of circulation Γ (see figure 3a), with azimuthal velocity

$$v_\theta = \frac{\Gamma}{2\pi r}. \quad (3.1)$$

We first describe the kinematics of the blob deformation. A fluid particle in the blob located at a distance r from the centre of the vortex turns during time t by an angle θ :

$$\theta(r, t) = \int_0^t \frac{v_\theta}{r} dt = \frac{\Gamma t}{2\pi r^2}. \quad (3.2)$$

A scalar strip of initial length dr , located at a distance r from the vortex centre (figure 3a) is stretched so that its length at time t equals

$$dX = \sqrt{dr^2 + (r d\theta)^2} = dr \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} = dr \sqrt{1 + \frac{\Gamma^2 t^2}{\pi^2 r^4}}. \quad (3.3)$$

Meanwhile, the transverse, or striation thickness $s(t)$ of the strip, in the absence of diffusion, decreases so that the surface $s(t)dX$ remains constant in this

two-dimensional flow:

$$s(t) = \frac{s_0 dr}{dX} = \frac{s_0}{\sqrt{1 + \Gamma^2 t^2 / (\pi^2 r^4)}}. \quad (3.4)$$

We now describe the scalar dissipation of the blob. The displacement field results locally in compression perpendicular to the strip, and extension along the strip. It is convenient to introduce a frame of reference (O, X, Y) with the X -axis locally aligned with the spiral as shown on figure 3(b). In that frame, the velocity field is prescribed by the temporal evolution of the striation thickness $s(t)$ as

$$U = -\frac{X}{s} \frac{ds}{dt} \quad \text{and} \quad V = \frac{Y}{s} \frac{ds}{dt}. \quad (3.5)$$

The evolution equation for the dye concentration c is the convection–diffusion equation in the (X, Y) coordinates:

$$\frac{\partial c}{\partial t} + U \frac{\partial c}{\partial X} + V \frac{\partial c}{\partial Y} = D \left(\frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial Y^2} \right). \quad (3.6)$$

The magnitude of the ratio of the two convective terms $V \partial c / \partial Y$ and $U \partial c / \partial X$ is proportional to the strip aspect ratio $1 + (\Gamma^2 t^2) / (\pi^2 r^4)$: the concentration varies more slowly along the spiral than in its transverse direction for $\Gamma t / r^2 > 1$ so that (3.6) becomes

$$\frac{\partial c}{\partial t} + \frac{Y}{s} \frac{ds}{dt} \frac{\partial c}{\partial Y} = D \frac{\partial^2 c}{\partial Y^2}. \quad (3.7)$$

A change of variables (see e.g. Ranz 1979; Marble 1988; Villiermaux & Rehab 2000) consisting in counting transverse distances in units of the striation thickness $s(t)$ and time in units of the current diffusion time $s(t)^2 / D$ transforms (3.7) into a simple diffusion equation with

$$\xi = \frac{Y}{s(t)} \quad \text{and} \quad \tau(r) = \int_0^t \frac{D dt'}{s(t')^2} = \frac{Dt}{s_0^2} + \frac{D \Gamma^2 t^3}{3 \pi^2 r^4 s_0^2} \quad \text{giving} \quad \frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2}. \quad (3.8)$$

If c_0 is the initial concentration of the dye, the initial conditions at $\tau = 0$ are

$$\left. \begin{aligned} c &= c_0 & \text{for } |\xi| < 1/2, \\ c &= 0 & \text{for } |\xi| > 1/2. \end{aligned} \right\} \quad (3.9)$$

The concentration profile at any time and radial position along the spiral is

$$c(\xi, \tau) = \frac{c_0}{2} \left[\operatorname{erf} \left(\frac{\xi + 1/2}{2\sqrt{\tau}} \right) - \operatorname{erf} \left(\frac{\xi - 1/2}{2\sqrt{\tau}} \right) \right]. \quad (3.10)$$

The maximal concentration is obtained at the profile centre $\xi = 0$:

$$c_M(r, t) = c_0 \operatorname{erf} \left(\frac{1}{4\sqrt{\tau}} \right) = c_0 \operatorname{erf} \left(\frac{1}{4\sqrt{Dt/s_0^2 + D\Gamma^2 t^3 / (3\pi^2 r^4 s_0^2)}} \right). \quad (3.11)$$

This relation can be examined from the experiment ($\Gamma = 14.2 \text{ cm}^2 \text{ s}^{-1}$, $D = 5 \times 10^{-6} \text{ cm}^2 \text{ s}^{-1}$ and $s_0 \approx 0.22 \text{ cm}$). Figure 4(a) shows the maximal dye concentrations as

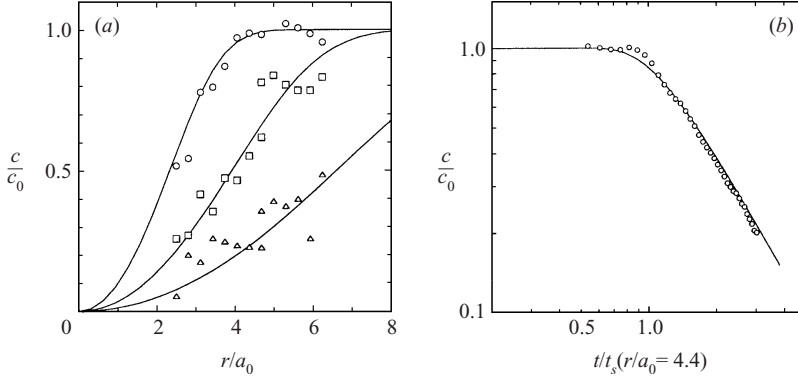


FIGURE 4. Comparison of the maximal dye concentrations obtained experimentally (symbols) and theoretically by (3.11) (solid lines). (a) Radial dependence at $t = 5$ s (\circ), $t = 10$ s (\square) and $t = 20$ s (\triangle). (b) Temporal dependence for $r/a_0 = 4.4$.

a function of the radius r at a fixed time, for three different times. The concentration falls to zero more rapidly closer to the spiral centre since the rate of elongation is higher there (see (3.3)).

Conversely, the temporal evolution of the concentration at a fixed r -location is constant (figure 4b) up to the mixing time $t_s(r)$. This makes the argument of the error function in (3.11) of order unity, i.e. $\tau = O(1)$:

$$t_s(r) = \frac{r^2}{\Gamma} \left(\frac{3\pi^2}{16} \right)^{1/3} \left(\frac{s_0}{r} \right)^{2/3} \left(\frac{\Gamma}{D} \right)^{1/3} \quad (3.12)$$

and displays the expected Péclet number dependence $Pe^{1/3}$, with $Pe = \Gamma/D$ characteristic of flows where material lines grow asymptotically linearly in time (see (3.3)). After the mixing time, the maximal concentration c_M decreases as $t^{-3/2}$, in close agreement with the trend shown on figure 4(b).

4. Probability density function

If A is the total surface area of the spiral with a non-zero concentration level, the probability density function (PDF) of the scalar $P(c)$ is the fraction of the total area with concentration lying in the interval $[c, c + dc]$. It is convenient to compute $P(c)$ in the (r, ξ) coordinates where ξ is defined in (3.8) so that with $dX = \sqrt{1 + (\Gamma^2 t^2)/(\pi^2 r^4)} dr$ and $dY = s d\xi = s_0 d\xi / \sqrt{1 + (\Gamma^2 t^2)/(\pi^2 r^4)}$, one has

$$P(c) dc = \iint_{c(X,Y) \in [c, c+dc]} \frac{dX dY}{A} = \iint_{c(r,\xi) \in [c, c+dc]} \frac{s_0 dr d\xi}{A}. \quad (4.1)$$

The scalar spatial distribution is given in (3.10) as the difference of two error functions. However, after the mixing time, that is when the spiral is very thin, this difference approximates the derivative of the error function, providing a Gaussian concentration profile:

$$c(\xi, r) = c_0 \operatorname{erf} \left(\frac{1}{4\sqrt{\tau(r)}} \right) e^{-\xi^2/2\sigma_\xi^2}, \quad (4.2)$$

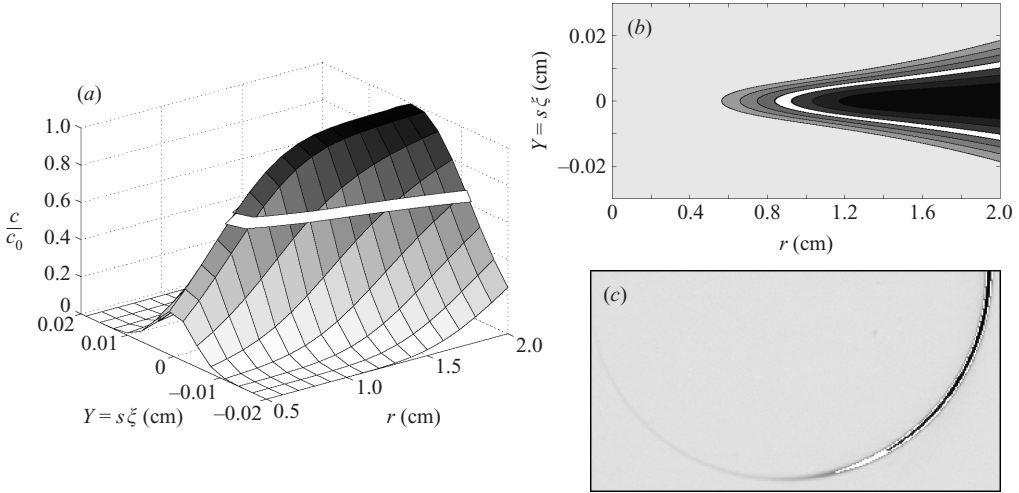


FIGURE 5. (a) Perspective view and (b) contour plot of the concentration profile given in (4.2). The white band corresponds to an iso-concentration $c/c_0 = 0.6$. (c) Zoom of the end of the spiral on figure 1 with the same construction.

where $\tau(r)$ is given by (3.8) and $\sigma_\xi(r)$ is the standard deviation of the original profile $c(Y)$ given in (3.10):

$$\sigma^2 = \frac{\int Y^2 c(Y) dY}{\int c(Y) dY} = s^2(t) \frac{\int \xi^2 c(\xi) d\xi}{\int c(\xi) d\xi} = s^2(t) \frac{1 + 24\tau(r)}{12} \quad \text{or} \quad \sigma_\xi^2 = \frac{1 + 24\tau(r)}{12}. \quad (4.3)$$

Note that the ‘spiral thickness’ σ first decreases as t^{-1} , reaches a minimum at $t = t_s$ and increases again as $t^{1/2}$ after the mixing time, when the spiral is locally nearly parallel to the vortex streamlines.

The shape of the iso-concentration lines $c(r, \xi) = c$ in the (r, ξ) -plane is shown in figure 5:

$$\xi(r, c) = \pm \sigma_\xi(r) \sqrt{2 \log[\operatorname{erf}(1/4\sqrt{\tau(r)})] - 2 \log(c/c_0)}. \quad (4.4)$$

This curve is defined for $r > r_1^*(c)$ only, that is above the smallest radius with the concentration c at time t :

$$r_1^*(c) = \left[\frac{16}{3\pi^2} \frac{D\Gamma^2 t^3}{s_0^2 [\operatorname{erf}^{-1}(c/c_0)]^{-2} - 16Dt} \right]^{1/4}. \quad (4.5)$$

If the scalar blob was initially placed between the radii r_1 and r_2 , the concentration PDF is

$$P(c) = \frac{2s_0}{A} \int_{\max[r_1, r_1^*(c)]}^{r_2} \left| \frac{\partial c}{\partial \xi} \right|^{-1} dr. \quad (4.6)$$

The concentration profile across the spiral and the evolution of the maximal concentration along the spiral set the global PDF.

The above relation is compared on figure 6 with the experimental histograms recorded with a blob initially located between $r_1 = 1.65$ cm and $r_2 = 2.1$ cm. In the

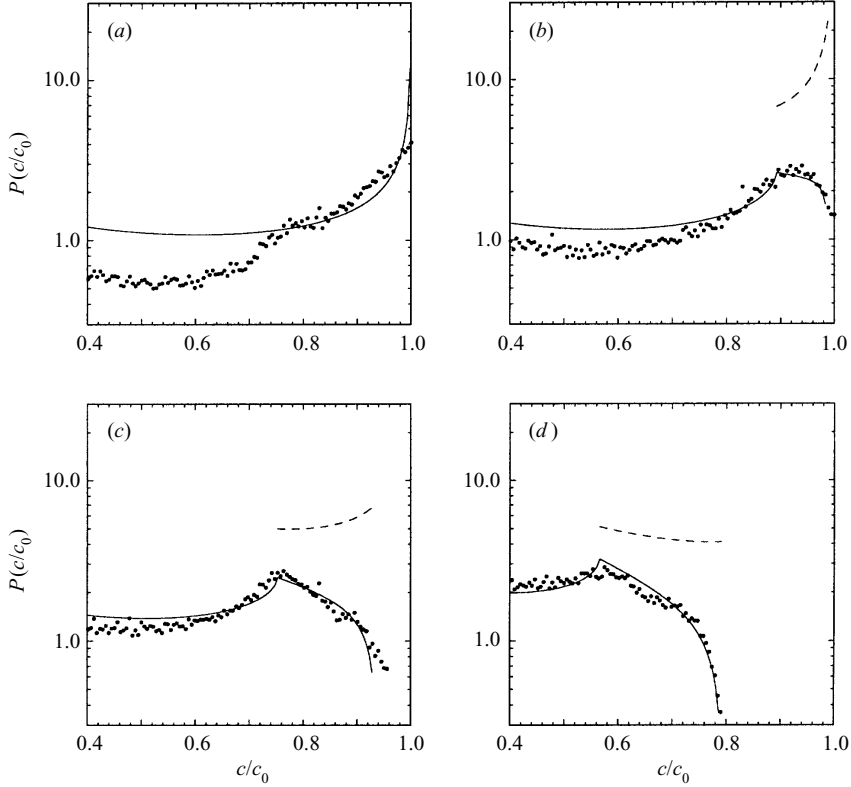


FIGURE 6. Probability density functions at (a) $t = 5$ s, (b) $t = 8$ s, (c) $t = 10$ s and (d) $t = 13$ s. Solid lines correspond to the theoretical prediction given by (4.6) and dashed lines correspond to the PDF of the spatial maxima of concentration, defined by (4.7).

early stages, (figure 6a), as long as most of the fluid particles constituting the spiral have not yet reached the mixing time, the PDF is that of a Gaussian spatial profile $(1/c)\sqrt{\log(c/c_M)}$ with $c_M = c_0$ displaying a characteristic \cup shape.

Once diffusion becomes effective, the PDF nucleates a cusp located at the maximal concentration $c_M(r_1)$ obtained at the inner end of the spiral. The shape of the PDF for $c_M(r_1) < c < c_M(r_2)$ results from the superposition of the right-hand branches of the \cup -shaped distributions parameterized by $c_M(r)$ with $r_1 < r < r_2$ (figure 6b–d) and weighted by the probability of finding the maximal concentration c_M , namely $Q(c_M)$. This distribution is the fraction of the spiral length dX with concentration in the interval $[c_M, c_M + dc_M]$:

$$Q(c_M) = \frac{1}{L} \left| \frac{dc_M}{dX} \right|^{-1}, \quad (4.7)$$

where L is the spiral length $L = \int_{r_1}^{r_2} dX$. It is defined in the range $[c_M(r_1), c_M(r_2)]$ and shown as the dashed line on figure 6. At short times, $P(c)$ and $Q(c_M)$ are very different because the low concentration levels at a small radii r and $\xi = 0$ are as numerous as the same levels at the edges of the Gaussian transverse profile ($\xi \neq 0$) at a higher r . The spatial distribution $c(\xi)$ contaminates the whole distribution $P(c)$, inducing the characteristic \cup shape. At later stages (figure 6d), the low levels of concentration from the edges of the Gaussian profile at large radii are sparse in comparison to those at

the centre of the spiral and $\xi = 0$. Therefore, $Q(c_M)$ becomes a decreasing function of c and gets closer to $P(c)$. In the final stages, when $\Gamma t/r^2 \gg 1$ and for $t_s(r) > 1$ for all r , these two distributions are both given by

$$P(c) \approx Q(c_M = c) \sim \left(\frac{\tilde{r}^4 s_0^2}{D\Gamma^2 t^3} \right)^{1/4} \frac{1}{c^{3/2}}, \tag{4.8}$$

where $\tilde{r} = (1/r_1 + 1/r_2)^{-1}$.

5. Conclusions and implications

In the simple displacement field of a two-dimensional vortex, a direct connection exists between the microscopic equations of diffusion and the resulting global statistics of the mixture through the scalar concentration PDF $P(c)$ which, therefore, appears as a reformulation of the microscopic convection–diffusion problem.

This one-to-one connection is possible because the flow solely causes a spatial mapping of the fluid particles with no interaction between the particles themselves. The concentration of a given fluid element evolves due to molecular diffusion and not because it interacts with a nearby element; indeed, the arms of the spiral never reconnect. This situation would lead to a completely different route for the evolution of $P(c)$. It is, in this respect, useful to learn that the distribution $Q(c_M)$ tends asymptotically towards $P(c)$, a hidden assumption made when considering mixture evolution by particle interaction (Curl 1963; Pope 1985; Pumir, Shraiman & Siggia 1991; Villiermaux 2002).

The simple stirring protocol considered here also provides an exact estimation of the scalar dissipation rate $\chi = -(d/dt)\langle c^2 \rangle = 2D\langle (\nabla c)^2 \rangle$, a quantity sometimes modelled in an *ad hoc* way. Here $\langle \cdot \rangle$ denotes a spatial integration, therefore

$$\chi = 2D \int_{r_1}^{r_2} \frac{dX}{s(t)} \int_{-\infty}^{+\infty} \left(\frac{\partial c}{\partial \xi} \right)^2 d\xi. \tag{5.1}$$

With $c(\xi)$ given in (3.10) and

$$\int_{-\infty}^{+\infty} \left(\frac{\partial c}{\partial \xi} \right)^2 d\xi \sim \frac{1 - e^{-1/8\tau(r)}}{\sqrt{\tau(r)}},$$

one sees that as soon as $\Gamma t/r^2 > 1$

$$\left. \begin{aligned} \chi &\sim (\Gamma/s_0)\sqrt{Dt} && \text{when } t < t_s(r) \text{ for all } r, \\ \chi &\sim s_0/(\sqrt{D}\Gamma)t^{-5/2} && \text{when } t > t_s(r) \text{ for all } r. \end{aligned} \right\} \tag{5.2}$$

As long as most of the fluid particles in the spiral have not reached the mixing time (i.e. when $t < t_s(r)$ and $\tau(r) \ll 1$), χ reflects both the diffusive smoothing ($\sim 1/\sqrt{Dt}$) at the edges of the concentration profile $c(\xi)$ and the increase of the concentration support length ($\sim \Gamma t$). When the mixing time has been reached all along the spiral (i.e. when $t > t_s(r)$ and $\tau(r) > 1$), the maximal concentration c_M decays as $t^{-3/2}$, the profile thickness σ increases again by pure diffusion as $t^{1/2}$ and the spiral length still increases like Γt , thus, since $\chi \sim (c_M/\sigma)^2 \sigma \Gamma t$, providing the $t^{-5/2}$ time dependence in (5.2).

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