Zonal flows driven by libration in rotating spherical shells: the case of periodic characteristic paths

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This work investigates the weakly nonlinear dynamics of internal shear layers and the mean zonal flow induced by the longitudinal libration of an inner core within a spherical shell. Building on the work of He et al. (J. Fluid Mech., vol. 939, 2022, A3), which focused on linear dynamics, we adopt a similar setup to explore the nonlinear regime using both asymptotic theory and numerical simulations, with Ekman numbers as low as $E = 10^{-10}$. A specific forcing frequency of $\widehat{\omega} = \sqrt{2}\widehat{\Omega}$, where $\widehat{\Omega}$ denotes the rotation rate, is introduced to generate a closed rectangular path of characteristics for the inertial wave beam generated at the critical latitude. Our approach extends previous results by Le Dizès (J. Fluid Mech., vol. 899, 2020, A21) and reveals that nonlinear interactions are predominantly localized around regions where the wave beam reflects on the boundary. We derive specific scaling laws governing the width and amplitude of nonlinear interactions: the width scales as $E^{1/3}$, while the amplitude scales as $E^{-1/6}$ in general. However, near the rotating axis, where the singularity of the self-similar solution becomes more pronounced, the amplitude exhibits a scaling of $E^{-1/2}$. In addition, our study also examines the nonlinear interactions of beams which are governed by different scaling laws. Through comparison with numerical results, we validate the theoretical predictions of the asymptotic framework, observing good agreement as the Ekman number decreases.

1. Introduction

Mechanical forces such as libration, precession, and tides, which arise from gravitational interactions, are essential to generate complex fluid flows within astrophysical and geophysical bodies (Le Bars *et al.* 2015). In the subsurface oceans of celestial bodies such as Enceladus, such forcing initiates dynamical processes that are crucial to understanding the internal structures of these bodies (Noir *et al.* 2009; Thomas *et al.* 2016; Soderlund *et al.* 2024). The energy dissipation from libration-driven turbulence may provide heat sources that maintain these subsurface oceans, while libration has been proposed as a potential mechanism driving planetary dynamos (Le Bars *et al.* 2011; Wu & Roberts 2013; Reddy *et al.* 2018; Wilson & Kerswell 2018). In the Earth's ocean, the interaction of tidal flows and supercritical topographies is an important source for creating strong concentrated internal wave beams, which play a key role in tidal conversion (Smith & Young 2003; Balmforth & Peacock 2009; Echeverri & Peacock 2010).

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In rotating fluids, when the external forcing frequency is less than twice the rotation frequency, smooth inertial modes can be excited within containers where regular inertial modes exist (Greenspan 1968). However, the dynamics becomes more complex in geometries such as spherical shells, where regular inertial modes generally do not exist. Instead, inertial wave beams are generated at critical latitudes due to oscillatory viscous concentrated boundary layer singularities (Kerswell 1995; Rieutord & Valdettaro 1997). Historical studies have shown that such wave beams can be well described by self-similar solutions (Moore & Saffman 1969; Thomas & Stevenson 1972). These solutions have been adapted to more complex cases, such as those involving beam reflections or limit cycles known as attractors (Maas *et al.* 1997; Rieutord *et al.* 2001; Le Dizès & Le Bars 2017). He *et al.* (2022) expanded the work of Le Dizès & Le Bars (2017) to a closed domain such as the spherical shell by deriving the asymptotic explicit expression of the linear harmonic velocity as a sum of many self-similar beams.

Transitioning to nonlinear phenomena, the interaction among wave beams introduces complexities, notably when beams intersect or reflect. Such interactions can lead to the creation of meanflow and second-harmonic corrections. For example, when beams of the same frequency interact as a result of reflection, they generate significant mean zonal flow and harmonic effects (Tabaei & Akylas 2003; Tabaei *et al.* 2005; Peacock & Tabaei 2005). These nonlinear processes manifest themselves notably in the formation of mean zonal flows, which have been extensively documented through experimental and numerical studies (Tilgner 2007; Sauret & Le Dizès 2013; Favier *et al.* 2014). Previous studies have demonstrated that, in the absence of inertial waves, nonlinear interactions within viscous boundary layers can drive zonal flows. The characteristics of these flows are strongly influenced by the libration amplitude, while remaining largely independent of the Ekman number and depend on the cylindrical variable r only (Busse 2010; Sauret & Le Dizès 2013). The dominant contribution to the meanflow arises in the bulk, manifesting as an azimuthal flow that scales as ϵ^2, where ϵ denotes the small oscillation amplitude. However, when the libration frequency falls below twice the rotation rate, allowing for the excitation of inertial waves, the resulting interactions become more complex, influencing the structure and behavior of mean zonal flows. This intricate interplay remains a significant challenge in fluid dynamics research within enclosed domains, where wave reflections and interactions can alter fundamental fluid behaviours (Cébron *et al.* 2021).

Of particular interest to the present study, Tilgner (2007) explored the dynamics of meanflow within a rotating spherical shell under the influence of an oscillating tidal mode. His research revealed that Reynolds stresses are primarily concentrated along wave beam paths, with an intensity peak observed at reflection points and beam intersections. In a subsequent investigation of thin viscous beams reflecting on flat boundaries in rotating and stratified fluids, Le Dizès (2020) found that when the libration frequency ω is less than the rotation rate Ω ($\omega < \Omega$), the reflection process generates both a second-harmonic correction and a meanflow correction. The second-harmonic beam exhibits a larger amplitude compared to the meanflow correction throughout most of the domain, except within the local interaction region.

Le Dizès (2020) noted that the structure of the wave beam reflections, which have a characteristic width scaling as $E^{1/3}$, maintained self-similarity with a $O(E^{1/6})$ correction, where E is the Ekman number. For scenarios involving purely stratified or rotating fluids, a meanflow correction occurs with an amplitude of $\epsilon^2 E^{-1/6}$, except in cases where the boundary is either horizontal or vertical, which exhibit a localized meanflow correction with a distinct triple-layer structure, including a significant $O(E^{4/9})$ viscous layer. More recent studies by Lin & Noir (2021) focused on the numerical analysis of nonlinear meanflow in a spherical shell with libration forcing at the inner core boundary. They observed that the

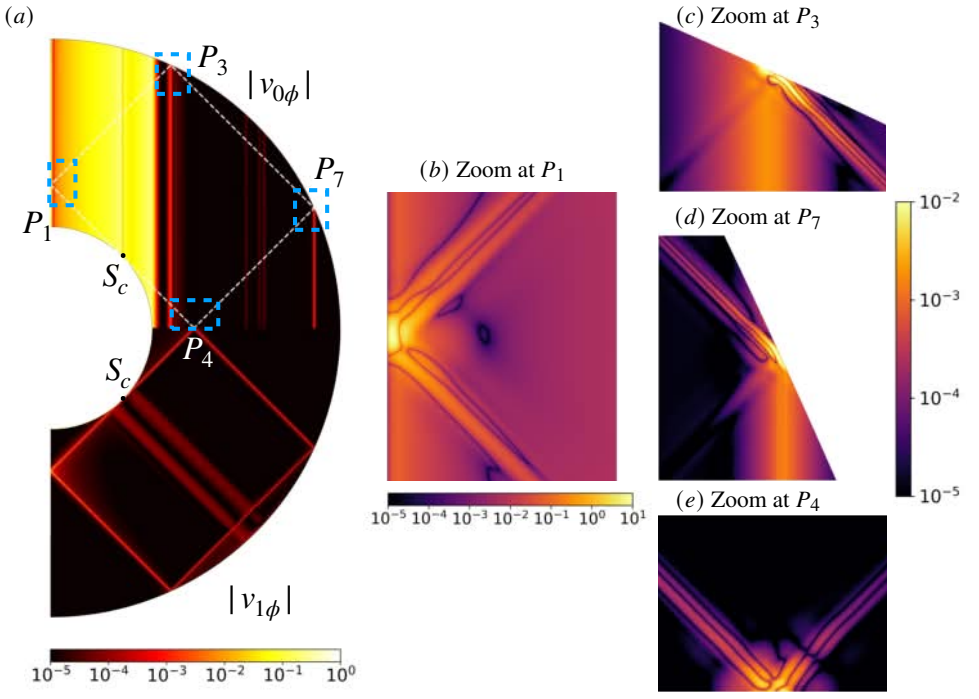

Figure 1: (a) Contours of the amplitudes of the meanflow azimuthal velocity $|v_{0\phi}|$ (upper part) and the linear harmonic solution $|v_{1\phi}|$ (lower part) for $E = 10^{-10}$. The critical latitude is marked at S_c whose cylindrical coordinates are $(\eta\sqrt{1-\omega^2/4}, \eta\omega/2)$. The aspect ratio of the spherical shell is $\eta = 0.35$, and the libration frequency of the inner core is $\omega = \sqrt{2}$. The white dashed lines represent the path of characteristics emitted from the critical latitude. (b)-(e) Zoomed-in regions of the interaction points located at P_1, P_3, P_7, and P_4, respectively.

meanflow correction remains localized at reflection points, scaling approximately as $E^{-1/6}$. Multiple bands were also identified in the bulk flow, directly corresponding to the positions of the reflection points. However, discrepancies were noted between theoretical predictions and numerical results, particularly regarding slight increases in the meanflow amplitude with the Ekman number.

In this paper, we extend the study to nonlinear meanflow within a rotating spherical shell subjected to libration forcing at the inner core boundary. Our objective is to use the linear self-similar solutions within the spherical shell to establish a theoretical framework for addressing meanflow corrections. This involves generalizing the previous results of Le Dizès (2020) and validating the scaling of the meanflow via numerical integration. For illustrative purposes, the internal shear layers in a spherical shell, induced by the libration of the inner core, along with the corresponding meanflow, are depicted in figure 1(a) at a low Ekman number $E = 10^{-10}$. Using direct numerical integration of the linear viscous governing equations, we first compute the linear harmonic velocity field. From this velocity field, we subsequently calculate the meanflow through a pseudo-spectral method. A detailed analysis of the interaction regions reveals that meanflow corrections are significantly larger in those regions than in both the outer region and the bulk bands, as shown in figures 1(b)-(e). These high-amplitude regions are of particular interest, motivating the development of an asymptotic theory to characterize the meanflow correction within these specific regions.

The structure of the paper is as follows: section 2 introduces the configuration and settings of the problem. Section 2.1 outlines the fundamental equations. In section 2.2, we review the

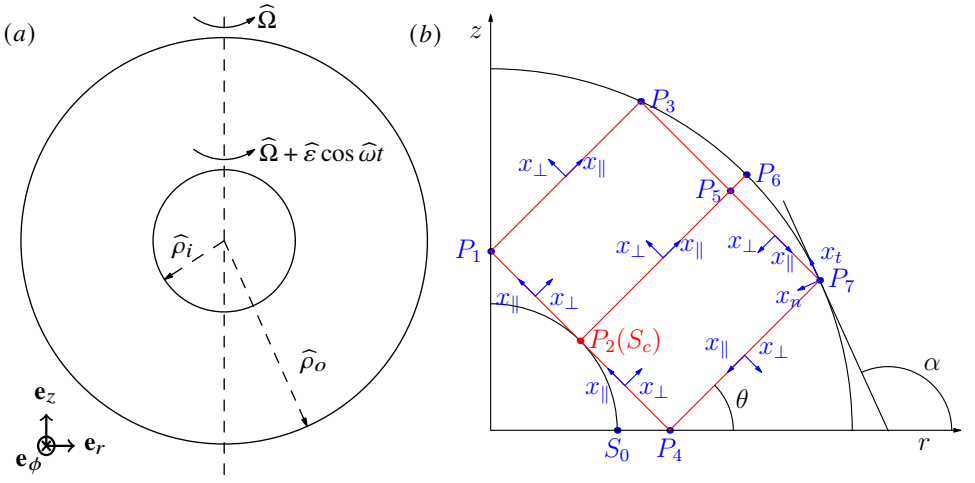

Figure 2: Sketch of the problem: (a) the outer radius of the concentric shell is $\widehat{\rho}_o$ and rotates at the angular velocity of $\widehat{\Omega}$, the inner radius is $\widehat{\rho}_i$ and rotates at an angular velocity of $\widehat{\Omega}+\widehat{\varepsilon}\cos\widehat{\omega}t$ with the amplitude $\widehat{\varepsilon}$ and frequency $\widehat{\omega}$ of the libration; (b) A local coordinate system is defined in the northward direction. The red lines represent the critical lines, while $x_\parallel$ and $x_\perp$ are the local coordinates. The southward direction is the inverse of the northward direction.

numerical methods employed to solve the linear harmonic governing equations and extend these approaches to address the nonlinear meanflow. Section 3 revisits the asymptotic theory, examining self-similar solutions and scaling laws. It also summarizes the structure of the linear harmonic solution for this configuration, providing approximations for this solution close to points where the interactions will be the strongest. The meanflow correction is addressed in section 4 and 5. We first compare numerical results with the predictions known for the contribution generated from oscillating boundary layers, in §4.1. We then consider the nonlinear interactions along the periodic beam, especially in the places where the interactions are the strongest, on the axis, and at the reflection points on the boundary (§4.2). Asymptotic expressions are derived and compared to numerical results. In §5, we focus on the meanflow band structures that originate from the strong interaction points. The scaling and form of the solution in these bands are analysed numerically using the asymptotic prescriptions that can be derived for these structures (appendix §C). Finally, the paper concludes with section 6, which summarizes the key findings and discusses their consequences for other configurations.

2. Framework

In this paper, we study the dynamics of an incompressible fluid with a constant kinematic viscosity ν, which fills a spherical shell and rotates around the axis $\mathbf{e}_z$ at a uniform rate $\widehat{\Omega}$. The flow is further subjected to the libration of the inner core, as depicted in figure 2(a). The inner core librates with a small amplitude $\widehat{\varepsilon}$ and at a frequency $\widehat{\omega}=\sqrt{2}\widehat{\Omega}$, resulting in an angular velocity of $\widehat{\varepsilon}\cos(\widehat{\omega}t)$ relative to the rotating system. Following the configuration of He *et al.* (2022), the radii of the inner and outer spherical cores are $\widehat{\rho}_i$ and $\widehat{\rho}_o$, respectively, with the aspect ratio $\eta=\widehat{\rho}_o/\widehat{\rho}_i=0.35$ to generate an Earth-like ratio. Time and space are non-dimensionalized using the rotation period $1/\widehat{\Omega}$ and the outer sphere radius $\widehat{\rho}_o$ of the spherical shell. The non-dimensional inner core radii is then η. The non-dimensional angular velocity of the outer and inner core are 1 and $1+\varepsilon\cos\omega t$ respectively, with libration amplitude $\varepsilon=\widehat{\varepsilon}/\widehat{\Omega}$ and frequency $\omega=\widehat{\omega}/\widehat{\Omega}$.

2.1. *Basic equations*

In the rotating frame, the velocity $\mathbf{V} = (v_r, v_\phi, v_z)$, written in cylindrical coordinates, and pressure P are governed by the following equations

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} + 2\mathbf{e}_z \times \mathbf{V} = -\nabla P + E\nabla^2\mathbf{V}\,, \tag{2.1a}$$

$$\nabla\cdot\mathbf{V} = 0\,, \tag{2.1b}$$

where E is the Ekman number, which is defined as $E = \nu/\widehat{\Omega}\widehat{\rho_o}^2$.

Contrarily to He *et al.* (2022), which exclusively addressed the linear aspects by omitting the advection term in equation (2.1*a*), this study expands into the nonlinear regime. We consider the following asymptotic expansion of the velocity $\mathbf{V}$ and pressure P:

$$(\mathbf{V}, P) = \varepsilon(\mathbf{v}_1, p_1)e^{-\mathrm{i}\omega t} + \varepsilon^2\left[(\mathbf{v}_0, p_0) + (\mathbf{v}_2, p_2)e^{-2\mathrm{i}\omega t}\right] + \text{c.c}, \tag{2.2}$$

where c.c denotes the complex conjugate, $(\mathbf{v}_1, p_1)$ is the linear harmonic, $(\mathbf{v}_0, p_0)$ is the meanflow correction, and $(\mathbf{v}_2, p_2)$ is the secondary harmonic velocity. In this paper, we focus solely on the meanflow correction.

Three steps are required to compute the meanflow correction. First, similarly to He *et al.* (2022), the linear response is found by solving

$$-\mathrm{i}\omega\mathbf{v}_1 + 2\mathbf{e}_z\times\mathbf{v}_1 = -\nabla p_1 + E\nabla^2\mathbf{v}_1\,, \tag{2.3a}$$

$$\nabla\cdot\mathbf{v}_1 = 0\,, \tag{2.3b}$$

$$\text{Inner boundary condition:}\quad \mathbf{v}_1 = r\mathbf{e}_\phi\,, \tag{2.3c}$$

$$\text{Outer boundary condition:}\quad \mathbf{v}_1 = \mathbf{0}\,. \tag{2.3d}$$

Once the linear harmonic velocity $\mathbf{v}_1$ is obtained, the nonlinear terms corresponding to its self-interaction, can be calculated at leading order. The resulting Reynolds stresses comprise two parts: the steady forcing and the second harmonic oscillating at 2ω. As the present study focuses on the meanflow, we only consider the steady Reynolds stress $\mathcal{N}_0$

$$\mathcal{N}_0 = \mathbf{v}_1\cdot\nabla\mathbf{v}_1^* + \text{c.c}\,, \tag{2.4}$$

where * denotes complex conjugation. The meanflow response $\mathbf{v}_0$ corresponds to a perturbation with a zero frequency $\omega = 0$ and is governed by a linear forced system:

$$2\mathbf{e}_z\times\mathbf{v}_0 + \nabla p_0 = -\mathcal{N}_0 + E\nabla^2\mathbf{v}_0\,, \tag{2.5a}$$

$$\nabla\cdot\mathbf{v}_0 = 0\,, \tag{2.5b}$$

$$\text{Inner and outer boundary condition:}\quad \mathbf{v}_0 = \mathbf{0}\,. \tag{2.5c}$$

where the forcing $\mathcal{N}_0$ arises from the nonlinear sefl-interaction of the linear harmonic v_1.

2.2. *Numerical approach*

To solve the equations outlined in section 2.1 and validate our theoretical predictions through asymptotic analysis, we have employed high-precision spectral methods for the numerical integrations. These methods, previously used in Rieutord & Valdettaro (1997) and He *et al.* (2022) to solve the linear governing equation (2.3), have enabled us to capture the linear harmonic velocity profiles $\mathbf{v}_1$ displayed in figure 1(a). We have looked for the solution of the linear governing equations (2.3*a*) expressed in vorticity form as

$$-\mathrm{i}\omega\nabla\times\mathbf{v}_1 + 2\nabla\times(\mathbf{e}_z\times\mathbf{v}_1) = E\nabla\times(\nabla^2\mathbf{v}_1)\,. \tag{2.6}$$

For this three-dimensional problem, we use spherical coordinates (ρ, θ, ϕ) representing radial, polar, and azimuthal directions, respectively. The velocity fields are expanded using

spherical harmonics in the polar and azimuthal directions and Chebyshev polynomials in the radial direction:

$$\mathbf{v}_1 = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} u_m^l(\rho)\mathbf{R}_l^m + v_m^l(\rho)\mathbf{S}_l^m + w_m^l(\rho)\mathbf{T}_l^m \,, \tag{2.7}$$

where,

$$\mathbf{R}_l^m = Y_l^m(\theta,\phi)\mathbf{e}_\rho, \quad \mathbf{S}_l^m = \nabla Y_l^m, \quad \mathbf{T}_l^m = \nabla \times \mathbf{R}_l^m \,. \tag{2.8}$$

Projecting the vorticity equation (2.6) onto this basis allows us to solve the linear system using a block tridiagonal algorithm. In our cases, we are assuming an axisymmetric fluid response along with the no-slip boundary conditions on the spherical shell. Detailed descriptions of the numerical methods and the construction of the system are available in He *et al.* (2022) and He *et al.* (2023) for readers interested in further details.

After obtaining the linear harmonic response numerically, we apply a pseudo-spectral method to calculate the steady Reynolds stress. Subsequently, we solve the governing equations (2.5) to determine the meanflow velocity, $\mathbf{v}_0$, shown in figure 1(a). The nonlinear vorticity equation for the mean zonal flow is expressed as

$$2\nabla \times (\mathbf{e}_z \times \mathbf{v}_0) = -\nabla \times (\mathbf{v}_1 \cdot \nabla \mathbf{v}_1^* + \text{c.c}) + E\nabla \times \nabla^2 \mathbf{v}_0 \,. \tag{2.9}$$

Details on the pseudo-spectral numerical workflow are provided in appendix A and illustrated in figure 24. We have also investigated potential aliasing issues in this nonlinear setup and found no significant impact on the results, regardless of whether dealiasing was applied or not. Consequently, all the results discussed in the present paper have not been de-aliased. The convergence of our simulations is further demonstrated in appendix A and the numerical resolutions for different Ekman numbers are documented in table 1 of this appendix. It is important to note that the resolutions required for the nonlinear simulations exceed those used in the purely linear analyses, to ensure similar convergence properties in both cases.

3. Asymptotic description of the harmonic solution

As first demonstrated by Moore & Saffman (1969), the propagation and viscous smoothing of a localized singularity can be described, in the limit of small Ekman numbers, by a self-similar solution. This viscous smoothing leads to the emergence of a self-similar expression for the primary components of the wave beam velocity. Note that the concentrated beams that originate from the critical latitude are linked to an inviscid singularity along the critical ray, as identified by Le Dizès (2024).

In the scenario under consideration, it is postulated that concentrated beam rays emanate from the critical latitude singularity where the boundary is locally tangent to the direction of inertial wave propagation, denoted as $S_c(r,z) = (\eta\sqrt{1-\omega^2/4}, \eta\omega/2)$, and propagate in two distinct directions: clockwise, designated as the northward direction, following the path $S_c \to P_1 \to P_3 \to P_7 \to P_4 \to S_c$, and counterclockwise, referred to as the southward direction. As shown in figure 2(b), P_1 is the point on the characteristic path crossing the rotation axis, P_3 and P_7 are the reflection points on the outer boundary while P_4 is on the equator where two beams are crossing. A local coordinate system $(x_\parallel, x_\perp)$ is introduced to locally describe the asymptotic structure of the beam, where $x_\parallel$ represents the distance from the source located at the critical latitude point S_c, and $x_\perp$ denotes the distance measured perpendicularly to the direction of beam propagation. For simplicity, it is assumed that the orientation of propagation does not change sign. When considering southward propagation, the direction of the corresponding local coordinate system is reversed.

In the following subsections, we provide expressions for the harmonic solution in the various regions where the meanflow correction becomes significant. We first review the results obtained by He *et al.* (2022) on the characteristic path, then present asymptotic expressions near the various interaction regions.

3.1. *Self-similar solution and scaling*

For the harmonic solution, the localized concentrated beams travel on the beam ray with a constant width of order $E^{1/3}$. The self-similar beam solution provided by Moore & Saffman (1969) was used by Le Dizès & Le Bars (2017) and He *et al.* (2022) to describe at leading order the velocity $v_{1\parallel}$ along the direction of the beam propagation,

$$v_{1\parallel} \sim \frac{1}{\sqrt{r}} C_0 H_m(x_\parallel, x_\perp) = \frac{1}{\sqrt{r}} C_0 \left(\frac{x_\parallel}{2\sin\theta}\right)^{-m/3} h_m(\zeta) \ . \tag{3.1}$$

The special function $h_m(\zeta)$ is,

$$h_m(\zeta) = \frac{e^{-\mathrm{i}m\pi/2}}{(m-1)!} \int_0^{+\infty} e^{\mathrm{i}p\zeta - p^3} p^{m-1} dp \ , \tag{3.2}$$

with the similarity variable,

$$\zeta = \frac{x_\perp}{E^{1/3}} \left(\frac{2\sin\theta}{x_\parallel}\right)^{1/3} \ . \tag{3.3}$$

θ is the angle between the inertial wave propagation direction and the equatorial plane, where $\theta = \pi/4$ enables the formation of a simple closed wave circuit in our configuration (see figure 2(b)). This angle also defines the inclination of the internal shear layers relative to the equatorial plane, with the libration frequency $\omega = 2\cos\theta$. The index m of the function $h_m(\zeta)$ characterizes the nature of the singularity that leads to the formation of the beam (Moore & Saffman 1969; Thomas & Stevenson 1972), while the complex amplitude C_0 represents the amplitude of the singularity. A brief review of different values of the index m for various configurations is provided by Le Dizès (2020). In particular, for any axisymmetric convex librating object, such as the inner sphere considered in the present study, Le Dizès & Le Bars (2017) determined the values $m = 5/4$ and C_0 by matching the similarity solution with the boundary layer solution around the critical latitude leading to

$$C_0^N = E^{1/12}\tilde{C}_0^N = E^{1/12} \frac{e^{\mathrm{i}\pi/2}}{8(2\sin\theta)^{3/4}}, \quad \text{northward ray} \ , \tag{3.4a}$$

$$C_0^S = E^{1/12}\tilde{C}_0^S = E^{1/12} \frac{e^{\mathrm{i}3\pi/4}}{8(2\sin\theta)^{3/4}}, \quad \text{southward ray} \ . \tag{3.4b}$$

Expression (3.1) describes the main velocity component. In the (r, z) plane, there also exists a component along $\mathbf{e}_\perp$ which is $O(E^{1/3})$ smaller.

As shown by equations (3.4), the velocity amplitude scales as $E^{1/12}$ in the asymptotic theory. Internal shear layers within a spherical shell were observed to preserve their self-similar structure upon reflecting on boundaries, undergoing either contraction or expansion during this process. Particular attention was given to how these layers propagate along a periodic rectangular path at specific frequencies of the libration forcing (He *et al.* 2022). Additionally, reflections on the rotating axis were found to induce a phase shift, facilitating the convergence of the summation describing the beam superpositions. The final expressions for the velocity components were derived from these self-similar solutions, resulting in the

following compact form in the (r, z) plane:

$$\mathbf{v}_{1_{2D}} \sim \frac{E^{1/12}}{\sqrt{r}}(\tilde{C}_0^N G_m(x_\perp^N, x_\parallel^N, L^N)\mathbf{e}_\parallel^N + \tilde{C}_0^S G_m(x_\perp^S, x_\parallel^S, L^S)\mathbf{e}_\parallel^S)\ . \tag{3.5}$$

where the new function G_m is given by

$$G_m(x_\perp, x_\parallel, L) = \left(\frac{x_\parallel}{2\sin\theta}\right)^{-m/3} g_m(\zeta, L/x_\parallel)\ , \tag{3.6a}$$

$$g_m(\zeta, L/x_\parallel) = \frac{e^{-\mathrm{i}m\pi/2}}{(m-1)!}\int_0^\infty \frac{e^{\mathrm{i}p\zeta - p^3} p^{m-1}}{1 - \mathrm{i}e^{-p^3 L/x_\parallel}} dp\ . \tag{3.6b}$$

Compared to the function h_m in (3.2), the denominator of g_m comes from the beam summation and the "i" factor from the phase shift induced by the reflection on the axis. Note that $\mathbf{e}_\parallel^S = -\mathbf{e}_\parallel^N$.

Expression (3.5) describes the solution on the rectangular circuit $P_1P_3P_7P_4$. As explained in He *et al.* (2022), a weaker secondary beam is also present between S_c and P_6 owing to the reflection on the inner core. The width scaling of this secondary beam is $E^{1/6}$, indicating a relatively larger transverse scale on which viscous diffusion becomes negligible. Consequently, the parallel velocity component $v_{1\parallel}$ of the beam remains constant throughout its path. Furthermore, the amplitude of the beam scales as $E^{1/4}$, which is significantly smaller than the amplitude of the solution on the main beam. The velocity profile along the beam is asymptotically described by

$$\mathbf{v}_{1_{2D}} \sim \frac{E^{1/4}}{\sqrt{r}}(\tilde{C}_0^N \breve{F}_m(\breve{x}_\perp^N, \lambda^N)\mathbf{e}_\parallel^N + \tilde{C}_0^S \breve{F}_m(\breve{x}_\perp^S, \lambda^S)\mathbf{e}_\parallel^S) \tag{3.7}$$

where

$$\breve{F}_m(\breve{x}_\perp, \lambda) = \frac{\breve{x}_\perp}{\sqrt{2}}\frac{e^{-\mathrm{i}m\pi/2}}{(m-1)!}\int_0^\infty e^{\mathrm{i}p\lambda(\breve{x}_\perp)^2}\frac{e^{-p^3L/\sqrt{2}}p^{m-1}}{1 - \mathrm{i}e^{-p^3L/\sqrt{2}}}dp\ , \tag{3.8}$$

with $\lambda^N = -\sqrt{2}/4$, $\lambda^S = \sqrt{2}/4$ and $\breve{x}_\perp = x_\perp/E^{1/6}$.

A detailed comparison between this asymptotic solution and numerical solutions of the linear harmonic viscous problem can be found in He *et al.* (2022).

The azimuthal velocity, $v_{1\phi}$, can be readily derived from $v_{1\parallel}$ using

$$v_{1\phi} \sim \pm\mathrm{i}v_{1\parallel}\ , \tag{3.9}$$

where the positive sign (resp. negative sign) corresponds to an obtuse angle (resp. acute angle) between $\mathbf{e}_\parallel$ and $\mathbf{e}_r$. This equation also implies, for $\theta = \pi/4$, that

$$v_{1\phi} \sim -\sqrt{2}\mathrm{i}v_{1r}. \tag{3.10}$$

The velocity field defined by (3.5) and (3.7) also satisfies, for $\theta = \pi/4$, the remarkable property

$$\frac{\partial v_{1r}}{\partial z} \sim -\frac{\partial v_{1z}}{\partial r}. \tag{3.11}$$

The symbol $\sim$ in equations (3.9), (3.10) and (3.11) has to be understood as denoting equality up to corrections of order $E^{1/3}$. We shall see below the importance of these equations for simplifying the expression of the Reynolds stress.

For detailed information about the local coordinate system relationships of northward and southward ray propagation paths, as well as more details about the different beam amplitudes, see table 2 and table 3 in appendix B. In the following, we shall often keep the parameter

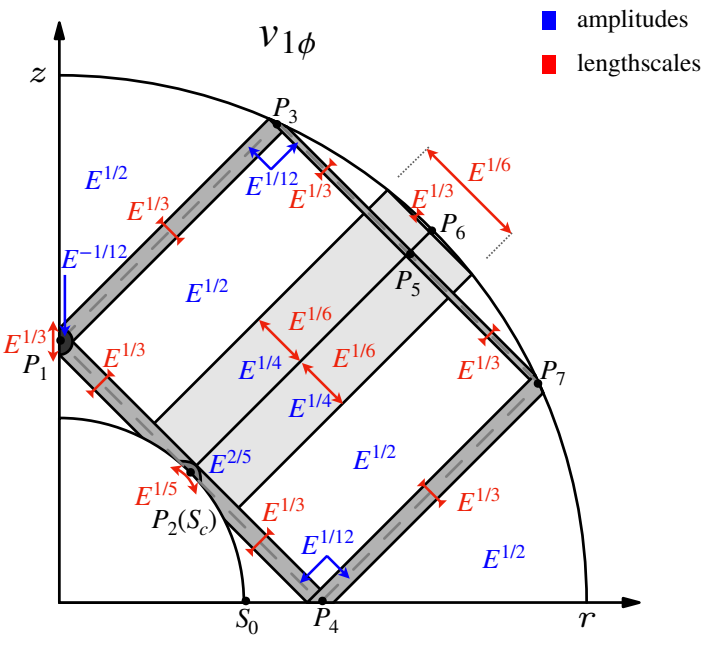

Figure 3: Ekman scalings of the azimuthal velocity $v_{1\phi}$ of the linear harmonic solution, reproduced from figure 4 in He *et al.* (2022), while the amplitudes and the lengthscales are shown in blue and red respectively.

m unprescribed to show the dependency of the various expressions with respect to this parameter, but one has to keep in mind that in the present analysis m and θ are actually fixed to $m = 5/4$ and $\theta = \pi/4$.

Figure 3 illustrates the scaling behaviour derived from the linear part of the asymptotic analysis. In this figure, the beam width scaling are shown in red, while the amplitude scaling are depicted in blue. In particular, the main beam is characterized by a beam width scaling like $E^{1/3}$ and an amplitude scaling like $E^{1/12}$. The corresponding scaling for the nonlinear meanflow will be discussed in the results section 5.

3.2. *Harmonic solution near P_3, P_4 and P_7*

The beams reflect on the outer boundary at the interaction points P_3 and P_7 while two beams intersect on the equatorial plane at the point P_4, see figure 2(b). For describing the solution near those points, it is convenient to introduce the local variables

$$\tilde{r}_\beta = \frac{r - r_{P_\beta}}{E^{1/3}}\ , \quad \tilde{z}_\beta = \frac{z - z_{P_\beta}}{E^{1/3}}, \tag{3.12}$$

and express the variables $x_\perp$ and $x_\parallel$ of the beam in terms of these local variables. Both northward and southward beams exhibit two parts, an incident beam, corresponding to the beam reaching P_β and a reflected beam, corresponding to the beam leaving P_β. Note that the point P_4 is not on a boundary, but the symmetry of the solution with respect to the equator makes the beam was reflecting on the horizontal plane. We shall use the compact notation $\tilde{B}^{Ni}_\beta, \tilde{B}^{Nr}_\beta, \tilde{B}^{Si}_\beta, \tilde{B}^{Sr}_\beta$ to denote the four contributions present near P_β. The northward incident beam reaching P_β is for instance given by

$$\tilde{B}^{Ni}_\beta(\tilde{r}_\beta, \tilde{z}_\beta) = \tilde{C}^{Ni}_{0\beta} G_m\left(x^{Ni}_{\perp\beta}, x^{Ni}_{\parallel\beta}, L^{Ni}_\beta\right)\ , \tag{3.13}$$

where the parameters $\tilde{C}_{0\beta}^{Ni}$ and L_{β}^{Ni} and the variables $x_{\perp\beta}^{Ni}$ and $x_{\parallel\beta}^{Ni}$ can be obtained using table 2 of Appendix B. For the southward beam, these quantities are given in table 3.

In each local region around P_3, P_4 and P_7, we can write the velocity field as

$$\mathbf{v}_1 \sim \frac{1}{\sqrt{r}} E^{1/12} \tilde{\mathbf{v}}_1 \,, \tag{3.14}$$

with

$$\tilde{v}_{1r} = 2^{-1/2}\tilde{B}_3^+ \,, \quad \tilde{v}_{1\phi} = -\mathrm{i}\tilde{B}_3^+ \,, \quad \tilde{v}_{1z} = 2^{-1/2}\tilde{B}_3^- \,, \quad \text{for } P_3, \tag{3.15a}$$

$$\tilde{v}_{1r} = -2^{-1/2}\tilde{B}_4^+ \,, \quad \tilde{v}_{1\phi} = \mathrm{i}\tilde{B}_4^+ \,, \quad \tilde{v}_{1z} = -2^{-1/2}\tilde{B}_4^- \,, \quad \text{for } P_4, \tag{3.15b}$$

$$\tilde{v}_{1r} = 2^{-1/2}\tilde{B}_7^- \,, \quad \tilde{v}_{1\phi} = -\mathrm{i}\tilde{B}_7^- \,, \quad \tilde{v}_{1z} = -2^{-1/2}\tilde{B}_7^+ \,, \quad \text{for } P_7, \tag{3.15c}$$

where

$$\tilde{B}_\beta^+(\tilde{r}_\beta, \tilde{z}_\beta) = (\tilde{B}_\beta^{Ni} - \tilde{B}_\beta^{Sr}) + (\tilde{B}_\beta^{Nr} - \tilde{B}_\beta^{Si}), \tag{3.16a}$$

$$\tilde{B}_\beta^-(\tilde{r}_\beta, \tilde{z}_\beta) = (\tilde{B}_\beta^{Ni} - \tilde{B}_\beta^{Sr}) - (\tilde{B}_\beta^{Nr} - \tilde{B}_\beta^{Si}). \tag{3.16b}$$

There is a relation between incident and reflected beams: they are such that the normal velocity of the sum of these two contributions vanish on the boundary. This condition can be written, at the three points P_3, P_4 and P_7, as

$$\tilde{B}_3^{Ni}(\tilde{x}_n = 0) = K\tilde{B}_3^{Nr}(\tilde{x}_n = 0) \;\;, \quad \tilde{B}_3^{Si}(\tilde{x}_n = 0) = (1/K)\tilde{B}_3^{Sr}(\tilde{x}_n = 0), \tag{3.17a}$$

$$\tilde{B}_4^{Ni}(\tilde{x}_n = 0) = \tilde{B}_4^{Nr}(\tilde{x}_n = 0) \;\;, \quad \tilde{B}_4^{Si}(\tilde{x}_n = 0) = \tilde{B}_4^{Sr}(\tilde{x}_n = 0), \tag{3.17b}$$

$$\tilde{B}_7^{Ni}(\tilde{x}_n = 0) = (1/K)\tilde{B}_7^{Nr}(\tilde{x}_n = 0) \;\;, \quad \tilde{B}_7^{Si}(\tilde{x}_n = 0) = K\tilde{B}_7^{Sr}(\tilde{x}_n = 0), \tag{3.17c}$$

where $\tilde{x}_n = 0$ denotes the boundary, and

$$K = \frac{\sin(\alpha + \pi/4)}{\sin(\alpha - \pi/4)} \tag{3.18}$$

the contraction factor of the northward beam at P_3.

However the tangential velocity does not vanish on the boundary. Consequently, in the local region P_3 and P_7 where a real boundary is present, a viscous boundary layer is expected to form. This boundary layer was studied in Le Dizès (2020). It was shown to induce a $O(E^{1/6})$ viscous correction to the reflected beam, which is further discussed below in section 3.5.

3.3. *Harmonic solution near P_5 and P_6*

The weak beam propagating between S_c and P_6 interacts with the main beam propagating between P_3 and P_7 at the point P_5. In this interaction region which is $O(E^{1/3})$ large and $O(E^{1/6})$ long in the direction aligned with the main beam, the harmonic solution exhibits a particular approximation that can be written as

$$\mathbf{v}_1 \sim \frac{1}{\sqrt{r}} (E^{1/4}\breve{\mathbf{v}}_1 + E^{1/12}\tilde{\mathbf{v}}_1) \,, \tag{3.19}$$

with

$$\breve{v}_{1r} = \frac{1}{\sqrt{2}}(\breve{B}^N - \breve{B}^S), \quad \breve{v}_{1\phi} = -\mathrm{i}(\breve{B}^N - \breve{B}^S), \quad \breve{v}_{1z} = \frac{1}{\sqrt{2}}(\breve{B}^N - \breve{B}^S), \tag{3.20a}$$

$$\tilde{v}_{1r} = \frac{1}{\sqrt{2}}(\tilde{B}_5^N - \tilde{B}_5^S), \quad \tilde{v}_{1\phi} = -\mathrm{i}(\tilde{B}_5^N - \tilde{B}_5^S), \quad \tilde{v}_{1z} = -\frac{1}{\sqrt{2}}(\tilde{B}_5^N - \tilde{B}_5^S) \,, \tag{3.20b}$$

where the beam structures are given for example for the northward beams by

$$\breve{B}^N = \tilde{C}_0^N \breve{F}_m(\breve{x}_\perp^N, \lambda^N) \,, \tag{3.21a}$$

$$\tilde{B}_5^N = \tilde{C}_{05}^N G_m(\tilde{x}_{\perp 5}^N, x_{\parallel 5}^N, L_5^N) \,, \tag{3.21b}$$

A similar expression is obtained for P_6, upon changing $\tilde{B}_5^N$ by $\tilde{B}_6^S$ and, $\tilde{B}_5^S$ by $\tilde{B}_6^N$. However, P_6 being on the boundary, a viscous correction is expected. Because P_6 is a critical point, this viscous correction is larger than that of P_3 and P_7. It is expected to be of order $E^{1/12}$ as shown in Le Dizès (2024). No such correction is generated at P_5.

3.4. *Harmonic solution near P_1*

The point P_1 where the critical beam reaches the axis is particular. At that point, the approximation (3.5) exhibits a singularity, which means that another approximation must be used in the local region near P_1. In that region, the velocity and pressure should be expressed using the Hankel transform, as discussed by Le Dizès & Le Bars (2017) (equation (A1) in appendix) and He *et al.* (2022) (equation (4.1) in their paper).

In Le Dizès (2015), the solution was derived for an open-domain configuration, focusing on the local region of beam reflection along the axis. The theory was extended to a closed domain with periodic characteristics in He *et al.* (2022). Introducing the local variables $\tilde{r}_1 = r/E^{1/3}$ and $\tilde{z}_1 = (z - \sqrt{2}\eta)/E^{1/3}$, they showed that the linear harmonic velocity in P_1 region can be written as

$$\mathbf{v}_1 \sim E^{-1/12} \hat{\mathbf{v}}_1 \,, \tag{3.22}$$

where

$$\hat{v}_{1r} = \frac{\mathrm{i}}{\sqrt{2}} \left(Q_1^N + Q_1^S \right) \,, \tag{3.23a}$$

$$\hat{v}_{1\phi} = Q_1^N + Q_1^S \,, \tag{3.23b}$$

$$\hat{v}_{1z} = \frac{1}{\sqrt{2}} \left(-Q_0^N + Q_0^S \right) \,. \tag{3.23c}$$

The functions $Q_l^N(\tilde{r}_1, \tilde{z}_1)$ and $Q_l^S(\tilde{r}_1, \tilde{z}_1)$ are defined, for $l = 0, 1$, by

$$Q_l^N(\tilde{r}_1, \tilde{z}_1) = \int_0^\infty \hat{V}^N(\tilde{k}) J_l(\tilde{k}\tilde{r}_1) e^{\mathrm{i}\tilde{k}\tilde{z}_1} \, d\tilde{k} \,, \tag{3.24a}$$

$$Q_l^S(\tilde{r}_1, \tilde{z}_1) = \int_0^\infty \hat{V}^S(\tilde{k}) J_l(\tilde{k}\tilde{r}_1) e^{-\mathrm{i}\tilde{k}\tilde{z}_1} \, d\tilde{k} \,, \tag{3.24b}$$

where J_l are Bessel functions of the first kind. The superscripts N and S designate northward and southward beam respectively.

The functions $\hat{V}^N$ and $\hat{V}^S$ are obtained by matching the Hankel transform expression to the solution (3.5) valid away from P_1. As shown He *et al.* (2022), this leads to the following expressions:

$$\hat{V}^N = 2^{m/2-3+1/8} \sqrt{\pi} e^{-\mathrm{i}\pi/4} \frac{e^{-\mathrm{i}m\pi/2}}{(m-1)!} \frac{\tilde{k}^{m-1/2} e^{-2x_\parallel^N \tilde{k}^3}}{1 - \mathrm{i} e^{-2L\tilde{k}^3}}, \tag{3.25}$$

$$\hat{V}^S = 2^{m/2-3+1/8} \sqrt{\pi} \frac{e^{-\mathrm{i}m\pi/2}}{(m-1)!} \frac{\tilde{k}^{m-1/2} e^{-2x_\parallel^S \tilde{k}^3}}{1 - \mathrm{i} e^{-2L\tilde{k}^3}}, \tag{3.26}$$

where we have used expressions (3.4) for C_0^N and C_0^S to simplify the expressions given in He *et al.* (2022), eq. (4.8a,b).

Note the scaling of the amplitude in $E^{-1/12}$ and of the region size in $E^{1/3}$, which has been illustrated in figure 3. The methodology developed for the P_1 region can be generalized to any beam interaction on the rotating axis. Note in particular that the functions $\hat{V}^N$ and $\hat{V}^S$ in our current analysis contain a term in $1 - \mathrm{i}e^{-2Lk^3}$ in the denominator, which arises from the summation of infinitely many beam contributions owing to the periodicity of the critical path. For a simple beam interaction without summation, no such term in the denominator is present (see Le Dizès 2015).

3.5. *Higher-order corrections to the harmonic solution*

In the previous sections, we have provided the leading-order approximation of the harmonic response in the interaction regions. These expressions are based on the similarity solution (3.1) obtained by Moore & Saffman (1969), which is known to be valid in an open domain up to $O(E^{1/3})$ corrections induced by variations along the beam.

Larger viscous corrections are created when the beam reflects on the boundary. Le Dizès (2020) showed that a viscous correction with an amplitude $O(E^{1/6})$ smaller than the main beam is created at reflection. He further showed that it takes the form of equation (3.1) with a larger index, $m+1$. In the present closed geometry, these corrected waves propagate on the closed circuit and, upon summation, yield an expression of the form (3.5) with an amplitude of order $E^{1/4}$ and functions G_{m+1} replacing G_m. Corrections of this form are expected to be created at the points P_3 and P_7 for both northward and southward beams.

For the reflections at P_2 and P_6, an even larger viscous correction is expected. Le Dizès (2024) indeed showed that a corrected beam of amplitude $O(E^{1/6})$ (that is $O(E^{1/12})$ smaller the main beam), was created at the critical point. This beam has also the similarity form (3.1) but an index $m + 5/4$. As for the reflections at P_3 and P_7, this correction is expected to accumulate as it propagates along the closed circuit, resulting in an expression of the form (3.5) with an amplitude of order $E^{1/6}$ and functions $G_{m+5/4}$ instead of G_m. Naturally, these first order corrections will themselves generate higher corrections of order $E^{1/4}$ as they reflect at P_2 and P_6, and of order $E^{3/12}$ as they reflect at P_3 and P_7. As a result, we expect the higher-order corrections to the harmonic solution (3.5) to be composed, up to $O(E^{5/12})$, of sum of functions $G^N_{m'}$ and $G^S_{m'}$ with different values m'. This has an important consequence: the harmonic solution including these viscous corrections satisfies equations (3.10) and (3.11) up to $O(E^{5/12})$.

4. Meanflow corrections

Our investigation of meanflow corrections proceeds systematically through two main aspects: the global response driven by oscillating boundary layers, and the localized interactions at beam crossing regions. For each interaction region, we present detailed comparisons between numerical and asymptotic solutions, accompanied by rigorous analysis of their scaling behaviors with respect to the Ekman number. This analysis encompasses the scaling of beam characteristics and velocity amplitudes.

To facilitate a clear visualization of the analysis locations, we employ the color-coded cutting system shown in figure 4. Blue markers indicate the locations where theoretical solutions are obtained and compared to numerical solutions, red markers designate the positions where only numerical results are analysed, and green markers highlight the cuts along which bulk flow solutions are compared.

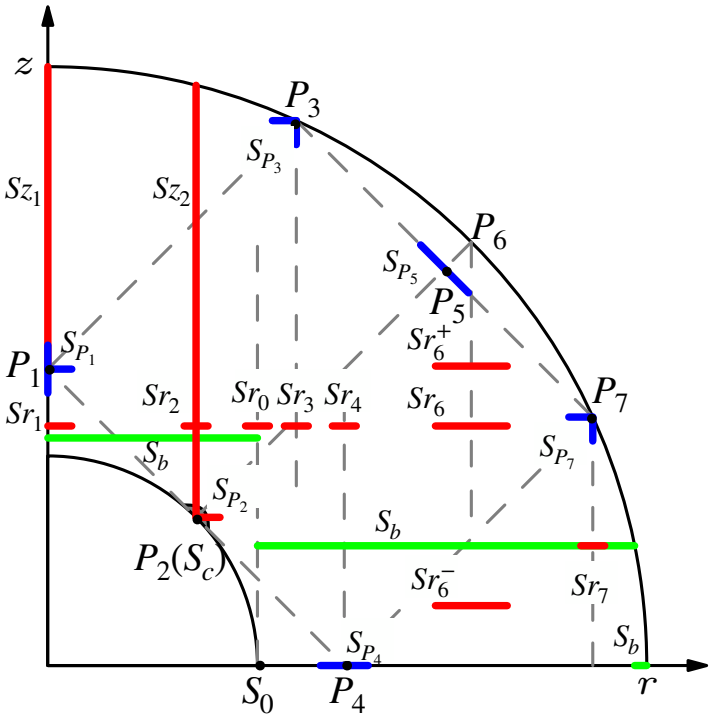

Figure 4: Schematic diagram of the positions of the slices used for comparing theoretical and numerical solutions (blue cuts). Green cuts are used to show the global bulk response. Red slices correspond to cases where only the numerical solution is discussed.

4.1. *Solution in the bulk generated from the boundary layers*

In this section, we consider the meanflow correction driven by oscillating boundary layers on the inner core. Such a correction was already examined by Sauret & Le Dizès (2013) in the absence of critical latitudes and inertial waves. They determined that the mean zonal flow within a spherical shell could be directly inferred from the analytical solutions derived for a full sphere. They showed that an azimuthal velocity of order E^0 was generated in the bulk, $v_{0\phi} = \mathcal{F}(\rho;\omega)$ in their equation (4.27). Furthermore, they extended their analysis to cases where $0 < \omega < 2$, a condition in which inertial waves emerge and have a weak nonlinear effect on the fluid. Building upon this foundational work, Cébron *et al.* (2021) expanded the theory to include non-homoeoidal spheroidal shells, accounting for variations in the amplitudes of libration forcing at different boundaries. They succinctly expressed the rotation velocity of the mean zonal flow in dimensionless form, presented as equation (5.5) in their publication.

In our study, we use equation (5.5) from Cébron *et al.* (2021), illustrating the theoretical predictions with a black dashed line in panel (a) of figure 5. Our numerical simulations at $E = 10^{-10}$ show excellent agreement with this theoretical prediction. This confirms that nonlinear effects within the Ekman boundary layer remain one of the important sources of mean zonal flow within the shell. Additionally, we identify two critical lines: $r = \eta/\sqrt{2}$, which marks the critical latitude, and $r = \eta$, denoting the cylinder tangent to the inner core boundary. Our numerical simulations confirm that equation (5.5) from Cébron *et al.* (2021) remains applicable even in the presence of inertial waves.

In the band issued from S_0, which is tangent to the inner core, we expect Stewartson layers characterized by three distinct widths: $E^{2/7}$ for the internal layers, $E^{1/3}$ for the inner layer, and

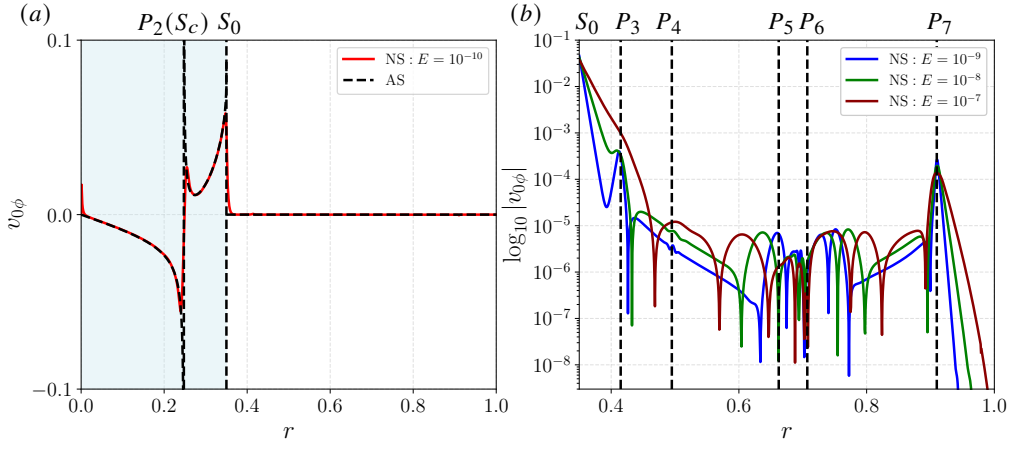

Figure 5: (a) Mean azimuthal velocity $v_{0\phi}$ in the bulk as predicted by numerical simulations (red solid line, $E = 10^{-10}$) and theoretical results (black dashed line, based on Sauret & Le Dizès (2013) and Cébron *et al.* (2021)). The light blue background highlights the interior region inside the tangent cylinder, located at $r = 0.35$. (b) Logarithm mean azimuthal velocity outside the tangent cylinder from numerical simulations at different Ekman numbers. For the range of cuts, in the tangent inner core, cuts are taken at S_b and $z = 0.4$. For the outer region, cuts are taken at $z = 0.2$ initially, and then on the equator. See figure 4 for the positions of the cuts. The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure5.

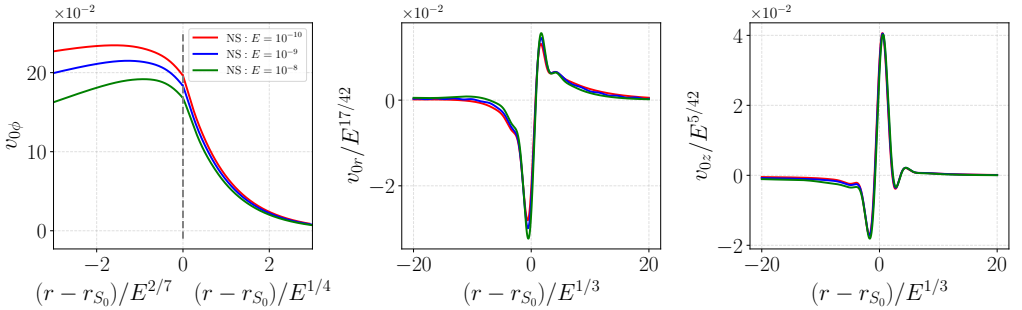

Figure 6: The length and amplitude scaling at Sr_0 are revealed by the velocity profiles of three Ekman numbers. The different scalings observed in the azimuthal velocity component $v_{0\phi}$ originate from the Stewartson layer.

$E^{1/4}$ for the external layers (Sauret & Le Dizès 2013). In figure 6(a), the azimuthal velocity is plotted using the internal and external layer variables. The external scaling is perfectly recovered for this component. For the radial and axial velocity components, the scalings in $E^{17/42}$ and $E^{5/42}$, that are predicted by the theory (Sauret & Le Dizès 2013) in the inner layer, are compatible with the numerical results, as shown in figures 6(b,c).

It is important to note that previous theoretical frameworks generally assumed that the velocity outside the tangent cylinder remains zero when forcing is applied at the inner core boundary only. Contrary to this assumption, our numerical results shown in figure 5(b) reveal that this is not the case when inertial waves are present. We indeed observe a bulk azimuthal flow of order $E^{1/3}$. Also visible are distinct peaks in the velocity profiles within the outer shell. These peaks correspond to specific regions where shear layers emitted from the critical latitude interact, either due to reflection on boundaries or through intersection within the volume. Although weak, these bands appear to exhibit the same scaling in E^0 as

the meanflow generated by the boundary layer on the inner core. This observation provides important context for the subsequent research questions: What is the origin of these weak nonlinear zonal beams? How can we determine the corresponding amplitude and beam width? Addressing these questions constitutes one of the objectives of the present study. In the following sections, we explain the origin of these weak nonlinear zonal beams and demonstrate the observed Ekman scaling.

4.2. *Solution generated from the interaction along the beam*

The equations (2.5) governing the meanflow correction read in cylindrical coordinates

$$-2v_{0\phi} + \frac{\partial p_0}{\partial r} = -\mathcal{N}_{0r} + E\left(\Delta - \frac{1}{r^2}\right) v_{0r}, \tag{4.1a}$$

$$2v_{0r} = -\mathcal{N}_{0\phi} + E\left(\Delta - \frac{1}{r^2}\right) v_{0\phi}, \tag{4.1b}$$

$$\frac{\partial p_0}{\partial z} = -\mathcal{N}_{0z} + E\Delta v_{0z}, \tag{4.1c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_{0r}) + \frac{\partial v_{0z}}{\partial z} = 0, \tag{4.1d}$$

where the Reynolds stress $\mathcal{N}_0 = (\mathcal{N}_{0r}, \mathcal{N}_{0\phi}, \mathcal{N}_{0z})$ is given by

$$\mathcal{N}_{0r} = v_{1r}\frac{\partial v_{1r}^*}{\partial r} + v_{1z}\frac{\partial v_{1r}^*}{\partial z} - \frac{v_{1\phi}v_{1\phi}^*}{r} + \text{c.c}\,, \tag{4.2a}$$

$$\mathcal{N}_{0\phi} = v_{1r}\frac{\partial v_{1\phi}^*}{\partial r} + v_{1z}\frac{\partial v_{1\phi}^*}{\partial z} + \frac{v_{1r}v_{1\phi}^*}{r} + \text{c.c}\,, \tag{4.2b}$$

$$\mathcal{N}_{0z} = v_{1r}\frac{\partial v_{1z}^*}{\partial r} + v_{1z}\frac{\partial v_{1z}^*}{\partial z} + \text{c.c}\,. \tag{4.2c}$$

The Reynolds stress expression can be simplified along the beam using the properties (3.10) and (3.11) that the harmonic solution satisfies. It gives using (3.10) and (3.11)

$$\mathcal{N}_{0r} \sim \frac{\partial}{\partial r}\left(|v_{1r}|^2 - |v_{1z}|^2\right) - 4\frac{|v_{1r}|^2}{r}\,, \tag{4.3a}$$

$$\mathcal{N}_{0\phi} \sim 2\frac{\partial}{\partial z}\,\mathrm{Im}(v_{1r}v_{1z}^*)\,, \tag{4.3b}$$

$$\mathcal{N}_{0z} \sim -\frac{\partial}{\partial z}\left(|v_{1r}|^2 - |v_{1z}|^2\right)\,. \tag{4.3c}$$

If we neglect the viscous terms, a particular solution to equations (4.1) can be obtained as

$$v_{0r} = -\frac{\partial}{\partial z}\,\mathrm{Im}(v_{1r}v_{1z}^*), \tag{4.4a}$$

$$v_{0\phi} = \frac{\partial}{\partial r}\left(|v_{1r}|^2 - |v_{1z}|^2\right) - 2\frac{|v_{1r}|^2}{r}, \tag{4.4b}$$

$$v_{0z} = \frac{1}{r}\frac{\partial}{\partial r} r\,\mathrm{Im}(v_{1r}v_{1z}^*). \tag{4.4c}$$

As the harmonic velocity $\mathbf{v}_1$ varies with respect to a spatial variable that scales as $E^{1/3}$ for the main beam, and as $E^{1/6}$ for the secondary beam, the viscous terms associated with the solution (4.4) remain small. This expression is therefore expected to be valid along the beam.

However, this solution is small, of order $E^{1/6}$, when evaluated using (3.1), and of order $E^{1/2}$, when evaluated using (3.7). It becomes important only when two beams interact, that is in the local regions P_β. In the next subsections, we provide the scaling and the expression of the solution in the three typical interaction regions.

4.2.1. *Solution in the local regions P_3, P_4 and P_7*

Close to those points, the particular solution reduces, at leading order, to

$$\mathbf{v}_0 \sim \frac{1}{r} E^{-1/6} \tilde{\mathbf{v}}_0 \,, \tag{4.5}$$

with

$$\begin{pmatrix} \tilde{v}_{0r} \\ \tilde{v}_{0\phi} \\ \tilde{v}_{0z} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} \dfrac{\partial \widetilde{\mathcal{N}}_{P_\beta}}{\partial \tilde{z}_\beta} \\ \dfrac{\partial \widetilde{\mathcal{M}}_{P_\beta}}{\partial \tilde{r}_\beta} \\ \dfrac{1}{\sqrt{2}} \dfrac{\partial \widetilde{\mathcal{N}}_{P_\beta}}{\partial \tilde{r}_\beta} \end{pmatrix}. \tag{4.6}$$

where the functions $\widetilde{\mathcal{M}}_{P_\beta}$ and $\widetilde{\mathcal{N}}_{P_\beta}$ are given by

$$\widetilde{\mathcal{M}}_{P_3} = 2\mathrm{Re}\widetilde{\mathcal{Q}}_3 \,, \quad \widetilde{\mathcal{M}}_{P_4} = 2\mathrm{Re}\widetilde{\mathcal{Q}}_4 \,, \quad \widetilde{\mathcal{M}}_{P_7} = -2\mathrm{Re}\widetilde{\mathcal{Q}}_7 \,, \tag{4.7a}$$

$$\widetilde{\mathcal{N}}_{P_3} = -2\mathrm{Im}\widetilde{\mathcal{Q}}_3 \,, \quad \widetilde{\mathcal{N}}_{P_4} = -2\mathrm{Im}\widetilde{\mathcal{Q}}_4 \,, \quad \widetilde{\mathcal{N}}_{P_7} = -2\mathrm{Im}\widetilde{\mathcal{Q}}_7 \,, \tag{4.7b}$$

with

$$\widetilde{\mathcal{Q}}_\beta(\tilde{r}_\beta, \tilde{z}_\beta) = (\tilde{B}_\beta^{Ni} - \tilde{B}_\beta^{Sr})(\tilde{B}_\beta^{Nr*} - \tilde{B}_\beta^{Si*}) \,. \tag{4.8}$$

As it can be seen on these expressions, only four nonlinear contributions remain. They are all confined to the local interaction regions. This result aligns with the outcomes of our numerical simulations and meets our expectations. This further supports the idea that the main sources of zonal flow are constrained to the $O(E^{1/3})$ regions where incident and reflected beams interact.

From (4.6), one can also obtain the normal and tangential velocity associated with this solution:

$$\tilde{v}_{0n} = -\frac{1}{\sqrt{2}} \frac{\partial \widetilde{\mathcal{N}}_{P_\beta}}{\partial \tilde{x}_t}, \quad \tilde{v}_{0t} = \frac{1}{\sqrt{2}} \frac{\partial \widetilde{\mathcal{N}}_{P_\beta}}{\partial \tilde{x}_n} \,. \tag{4.9}$$

Thanks to (3.17), one can see that expression (4.7) with (4.8) reduces at the boundary ($\tilde{x}_n = 0$) to $\widetilde{\mathcal{N}}_{P_\beta}(\tilde{x}_n = 0) = 0$. This implies that the normal velocity vanishes at the boundary. The non-penetration condition is then automatically satisfied. This has an important consequence. It means that the particular solution (4.6) is the leading-order solution in each local region around P_β. It is this expression that will be compared to the numerical solution close to each P_β. Since the functions $\widetilde{\mathcal{M}}_{P_\beta}$ and $\widetilde{\mathcal{N}}_{P_\beta}$ are both localized, all the velocity components are also localized near P_β at this order.

Figure 7 compares the profiles of three asymptotic velocity components with numerical solutions for three different Ekman numbers near P_7. The comparison is performed on the vertical and horizontal slices S_{P_7}, as depicted in figure 4. As expected, the asymptotic theoretical results increasingly align with the numerical data as the Ekman number decreases. However, discrepancies are observed in the radial velocity, v_{0r}, and the axial velocity, v_{0z}, near the boundary. These discrepancies arise because the numerical boundary condition employs a no-slip condition, causing the numerical solutions to approach zero at the boundary, while

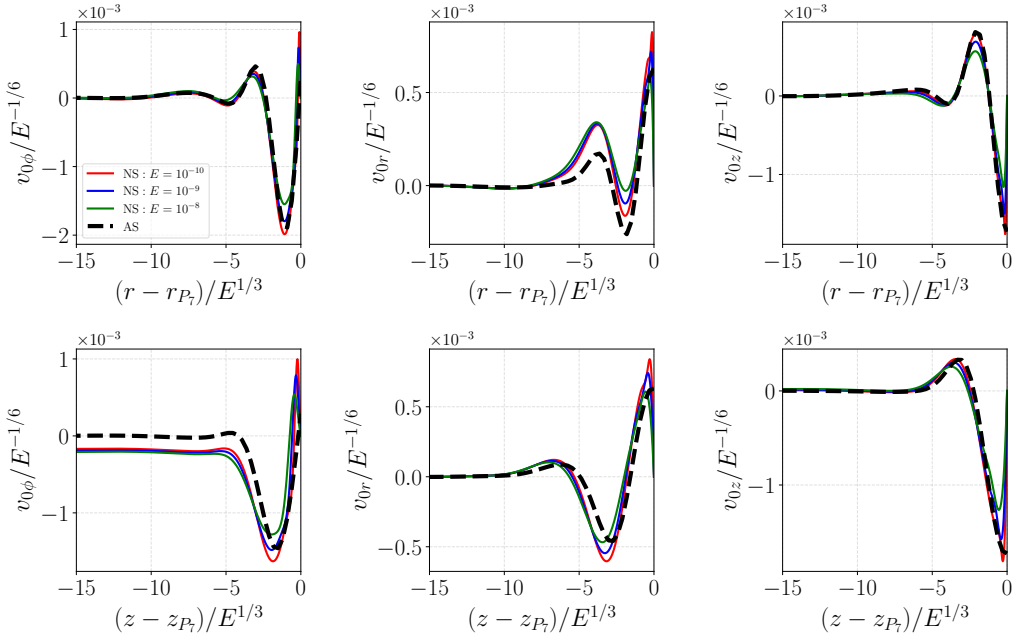

Figure 7: Three velocity components profiles of the asymptotic solution (*AS*) and the numerical solution for three different Ekman numbers at local region P_7 with two direction cuts (S_{P_7} in figure 4). The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure7.

only the non-penetration condition is satisfied by the asymptotic solution. As demonstrated in Le Dizès (2020), another approximation can be constructed in a viscous boundary layer of $O(E^{1/2})$ width to capture this behaviour.

In addition, the azimuthal velocity along the z-axis, parallel to the rotation axis, tends to zero in the asymptotic theory as one moves away from the local interaction point. However, numerical results display a small amplitude zonal flow in the bulk (see the bottom left panel of figure 7). This is also evident in figure 1, where several weak bands are visible throughout the bulk. The scaling of these bands will be discussed in the next section, using the results given in appendix C.

Comparisons between two similar interaction regions, P_3 and P_4, and the numerical results are presented in figures 8 and 9, respectively. The observations from figure 7 regarding differences in the boundary layer and the azimuthal velocity in the bulk are also applicable here. Notably, near P_4, located on the equatorial plane, the axial velocity v_{0z} remains zero by symmetry. Additionally, improvements in the comparison with decreasing Ekman numbers are observed, consistent with the findings at other local positions.

At the three specific locations, P_7, P_3, and P_4 the asymptotic analysis yielded consistent scaling behaviours: the interaction region scales as $E^{1/3}$, and the amplitude scales as $E^{-1/6}$ for all three velocity components. Note however that the rescaled velocity remains small, of order 10^{-3}, at each of these points. This explains why these regions are not more visible on the contour map shown in figure 1.

4.2.2. *Solution in the local regions P_5 and P_6*

The secondary weak beams encounter the main critical beam in the bulk at P_5. This case is interesting because the scalings of the two beams differ.

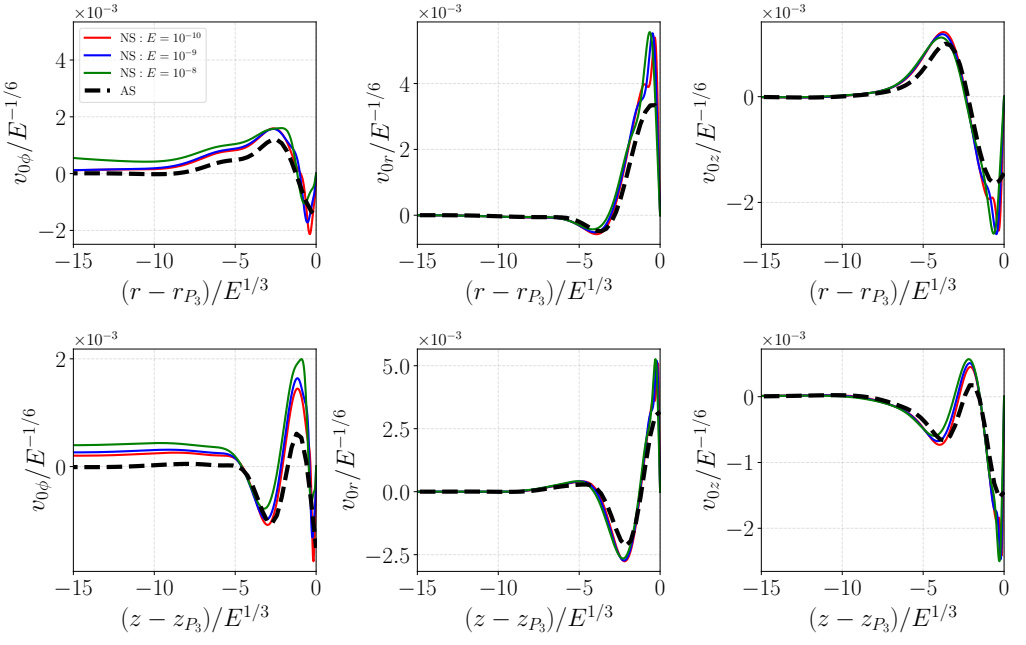

Figure 8: Same caption as figure 7 but at local region P_3 (Cuts at S_{P_3}). The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure8.

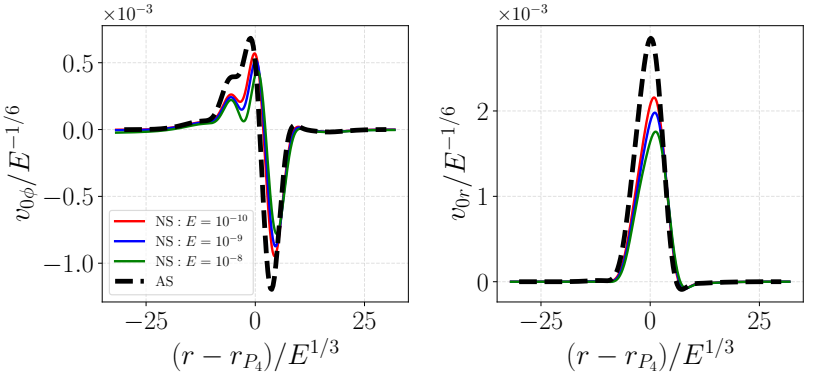

Figure 9: Comparison of asymptotic solutions and numerical results of two velocity components ($v_{0z} = 0$) at local region P_4. (Cuts position at S_{P_4}). The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure9.

The meanflow correction is weaker and given by

$$\mathbf{v}_0 \sim \frac{1}{r} E^0 \breve{\mathbf{v}}_0 \,, \tag{4.10}$$

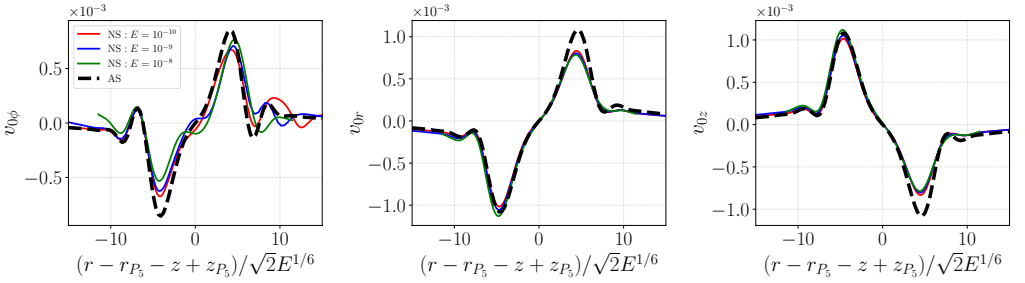

Figure 10: Comparison of the asymptotic solutions and numerical results of velocity profiles at local region P_5 (Cuts at S_{P_5}). The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure10.

with

$$\begin{pmatrix} \breve{v}_{0r} \\ \breve{v}_{0\phi} \\ \breve{v}_{0z} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sqrt{2}}\dfrac{\partial \breve{\mathcal{N}}_{P_5}}{\partial \tilde{z}_5} \\ \dfrac{\partial \breve{\mathcal{M}}_{P_5}}{\partial \tilde{r}_5} \\ \dfrac{1}{\sqrt{2}}\dfrac{\partial \breve{\mathcal{N}}_{P_5}}{\partial \tilde{r}_5} \end{pmatrix}, \tag{4.11}$$

where

$$\breve{\mathcal{M}}_{P_5} = 2\,\mathrm{Re}\left((\breve{B}^N - \breve{B}^S)(\tilde{B}_5^{N*} - \tilde{B}_5^{S*})\right), \tag{4.12}$$

$$\breve{\mathcal{N}}_{P_5} = -2\,\mathrm{Im}\left((\breve{B}^N - \breve{B}^S)(\tilde{B}_5^{N*} - \tilde{B}_5^{S*})\right). \tag{4.13}$$

These expressions are similar to the one obtained near point P_4, where the Nr and Si beams at P_4 are now the strong S and N beams propagating between P_3 and P_7, while Ni and Sr beams at P_4 are the weak S and N beams propagating between P_2 and P_6 (see figure 2(b)). Note that because $\breve{B}^S$ and $\breve{B}^N$ are functions of the slow variable $\breve{x}_\perp = E^{1/6}(\tilde{z}_5 - \tilde{r}_5)/\sqrt{2}$, the spatial derivatives in (4.11) should be applied to $\tilde{B}_5^S$ and $\tilde{B}_5^N$ only.

In figure 10, we compare the asymptotic solution from equation (4.11) with numerical results for three different Ekman numbers. The secondary weak beam, characterized by its larger width scaling in $E^{1/6}$, is analyzed by taking profile cuts parallel to the critical beam from P_3 to P_7 (see figure 4, S_{P_5}). Due to the anti-symmetry of the weak beam, the cuts in the perpendicular direction (that is on the $z = r$ line), are zero.

The width of the local region scales as $E^{1/6}$, while the other direction scales as $E^{1/3}$. The amplitude scaling is E^0, which is significantly weaker than in other local interaction regions. Despite the two beams having different scaling factors, the asymptotic method demonstrates overall accuracy. The asymptotic theory matches the numerical solutions well, exhibiting convergence as the Ekman number decreases.

However, some perturbations are observed on the right side of figure 10, specifically in the $v_{0\phi}$ component. These perturbations arise because the cut profiles are influenced by the weak band issued from the point P_6, as shown in figure 5(b).

In the local region P_6, a particular solution of the same form as (4.11) can be obtained. However, as we will see in §5 and Appendix C, this particular solution is modified by an additional homogeneous solution, which describes the meanflow correction associated with

the vertical band originating from P_6. Therefore, it cannot be directly compared to numerical results.

4.2.3. *Solution in the local region P_1*

For the point on the rotation axis, we must use the expressions (3.23) for the harmonic solution. We obtain the following expression for the meanflow

$$\mathbf{v}_0 = E^{-1/2}\hat{\mathbf{v}}_0 \,, \tag{4.14}$$

with

$$\hat{v}_{0r} = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial \tilde{z}_1}\left\{\mathrm{Re}\left[(Q_1^N + Q_1^S)(-Q_0^{N*} + Q_0^{S*})\right]\right\} \tag{4.15a}$$

$$\hat{v}_{0\phi} = \frac{1}{2}\frac{\partial}{\partial \tilde{r}_1}\left\{\left|Q_1^N + Q_1^S\right|^2 - \left|-Q_0^N + Q_0^S\right|^2\right\} - \frac{\left|Q_1^N + Q_1^S\right|^2}{\tilde{r}_1} \tag{4.15b}$$

$$\hat{v}_{0z} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \tilde{r}_1} + \frac{1}{\tilde{r}_1}\right)\left\{\mathrm{Re}\left[(Q_1^N + Q_1^S)(-Q_0^{N*} + Q_0^{S*})\right]\right\} \,. \tag{4.15c}$$

In figure 11, we compare the asymptotic solution from equation (4.15) with numerical results for three different Ekman numbers. The profiles analyzed are taken at S_{P_1} and include one profile in the direction perpendicular to the rotation axis and another along the rotation axis (see figure 4). Due to the axisymmetry of the system, only the axial velocity component is considered along the rotation axis, as the two other components, v_{0r} and $v_{0\phi}$, are identically zero.

The width of the local region scales with $E^{1/3}$, while the amplitude scales with $E^{-1/2}$. Although the overall agreement between the asymptotic solution and the numerical results is strong, particularly for the azimuthal component, a noticeable deviation is observed in the smaller v_{0r} component near the axis. This discrepancy was already observed in the linear harmonic solution in He *et al.* (2022). We suspect that it could come from higher-order corrections that could become non-negligible owing to the smallness of the solution at this leading order.

5. Numerical scalings of the bands

In the previous section, we analysed the meanflow corrections generated in the interaction regions and in the bulk from the oscillating boundary layer on the inner core. These represent the dominant contributions. As illustrated in figure 1, numerical results also reveal the presence of faint vertical beam-like bands within the bulk. These bands originate from the local interaction regions. However, the leading-order meanflow corrections computed in these regions were shown to be localized. The observed bands must therefore be associated with higher-order effects. Finding an approximation for the velocity field in the bands is thus also more complex, as it requires higher-order expansions of the harmonic solution. Nevertheless, some general insight into the scaling and structure of the velocity field in each band can be obtained through asymptotic reasoning. These results are summarized in Appendix C and will be used to interpret the numerical results that are now presented.

Figure 12 displays the scaling of the vertical band aligned with the rotating axis, analysed along the cut Sr_1 shown in figure 4. Near $r = 0$, the width of the band scales as $E^{1/3}$. As explained in the appendix, the amplitude of the axial and azimuthal velocity components in the band should be $E^{1/3}$ smaller than in the local region around P_1. This leads to expected scalings of $E^{-1/6}$ for $v_{0\phi}$ and v_{0z}, and $O(E^{1/6})$ for v_{0r}. These scaling are confirmed by the numerical results shown in figure 12. The variation of the meanflow correction within this

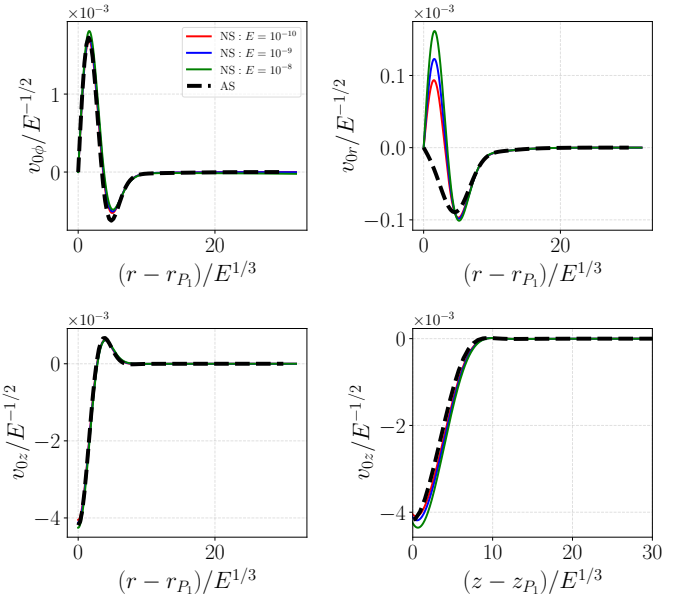

Figure 11: Comparison of the asymptotic solutions and numerical results of velocity profiles at local region P_1. (Cuts position at S_{P_1}). The directory containing the data and the Jupyter notebook used to compute the asymptotic theory and generate this figure can be accessed at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure11.

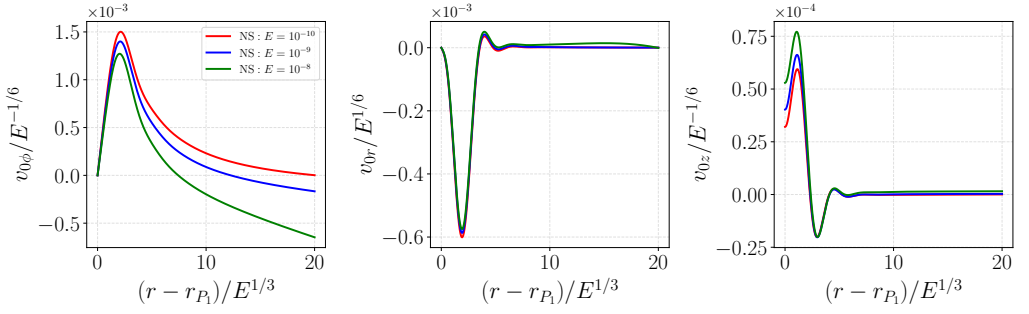

Figure 12: Velocity scalings in the band originating from P_1. Variation along the cut Sr_1 (indicated in figure 4) for three different Ekman numbers $E = 10^{-8}$, $E = 10^{-9}$ and E^{-10}. (a) Azimuthal velocity. (b) Radial velocity. (c) Axial velocity. The scalings are those predicted by the theory (appendix C).

band along the axial coordinate is also complex, as illustrated in figure 13(a).

Figure 14 illustrates the scaling behaviours in two distinct regions: the critical latitude region (analysed at cut S_{P_2}) and the resulting zonal band in the bulk (analysed at cut Sr_2). The horizontal width of the critical latitude region scales as $E^{2/5}$, while the width of the associated band is $O(E^{1/5})$. As explained in Appendix C, the scaling of the azimuthal velocity in the band depends on the Ekman pumping generated in the local region P_2, which is challenging to evaluate analytically. However, the scalings of the different components are interrelated. In figure 14(a), we propose a scaling in $E^{-1/10}$ for $v_{0\phi}$, $E^{1/2}$ for v_{0r} and $E^{1/5}$ for v_{0z} which is consistent with the expected structure of the meanflow correction in the band issued from P_2. Figure 14(b) present scaling behavior in the local region P_2 that may be compatible with the observed scalings in the band. In particular, a scaling of v_{0r} and v_{0z} in $E^{-1/5}$ is expected to yield an Ekman pumping of order $E^{1/5}$, which matches the amplitude of

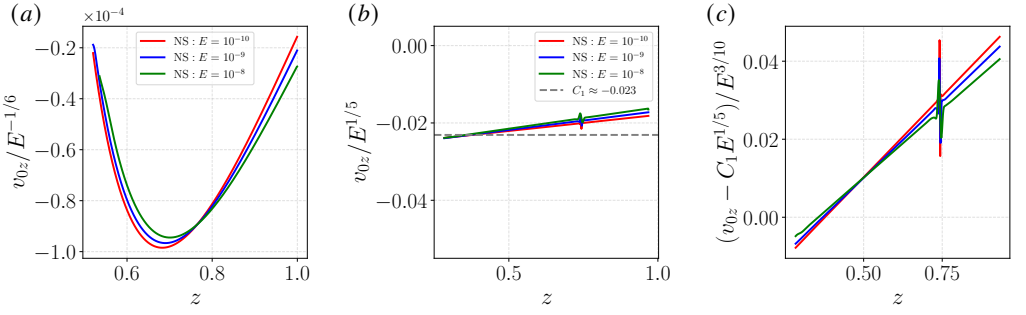

Figure 13: Vertical dependency of the bands originating from P_1 and P_2. (a): Band from P_1. Axial velocity along the axis (cut Sz_1), demonstrating a nonlinear dependency with respect to the vertical coordinate z; (b), (c): Band from P_2. Axial velocity versus z along the cut Sz_2, showing a uniform behavior at the order $E^{1/5}$ (b) and a linear variation at the order $E^{3/10}$. The localized variation of the velocity close to $z \approx 0.75$ in (b) and (c) corresponds to the crossing with the main beam travelling between P_1 and P_4.

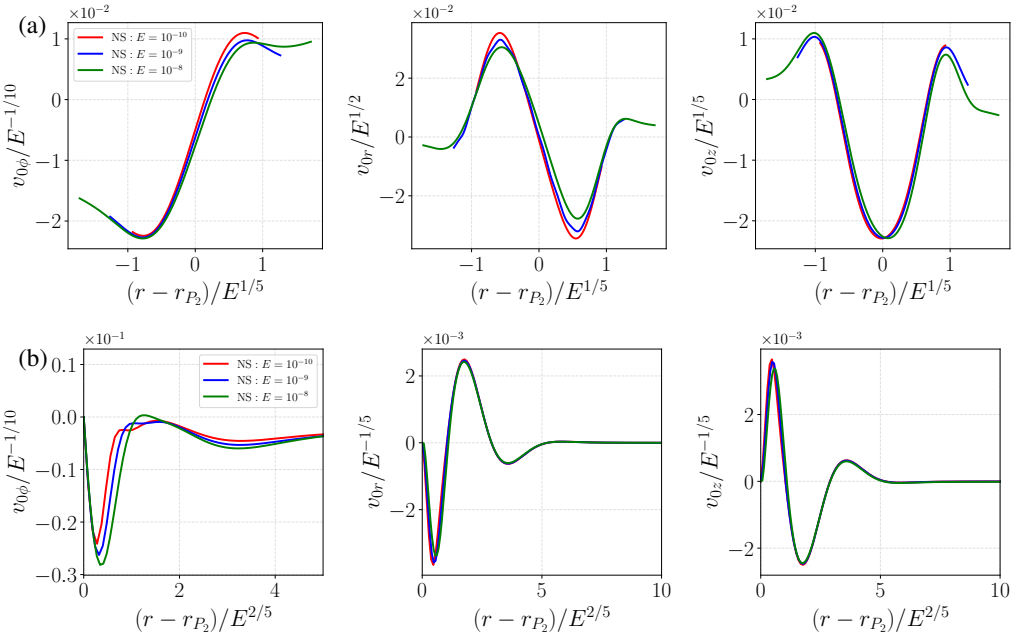

Figure 14: Velocity scalings in the local region P_2 (cut S_{P_2} in (b)) and in the band originating from P_2 (cut Sr_2 in (a)).

the axial velocity observed in the band. In figure 13(b,c), we observe that the axial velocity within the band from P_2 is uniform along the axial direction at leading order $E^{1/5}$, but linear at the next order $E^{3/10}$, as predicted by the theory.

Figures 15 and 16 present the velocity profiles along cuts Sr_3 and Sr_7, respectively. We have applied the scalings predicted by the asymptotic analysis, namely, an amplitude of order E^0 for the azimuthal and axial velocity components, and $E^{1/3}$ for the radial velocity (see appendix C). These scalings are consistent with the numerical results. However, for the azimuthal velocity in the band issued from P_7, an alternative scaling appears to better match the numerical data (see figure 17). At present, we have no theoretical justification for this alternate scaling.

As discussed in Appendix C, no significant band is expected to form from P_4, which

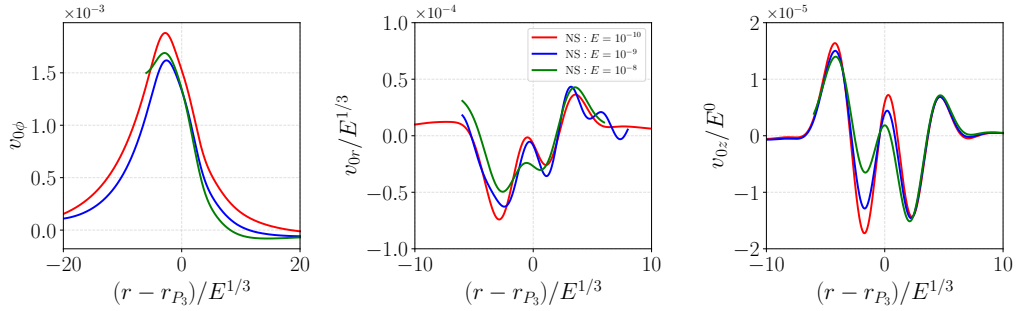

Figure 15: Same caption as figure 12 but for the band originating from P_3 (cut Sr_3).

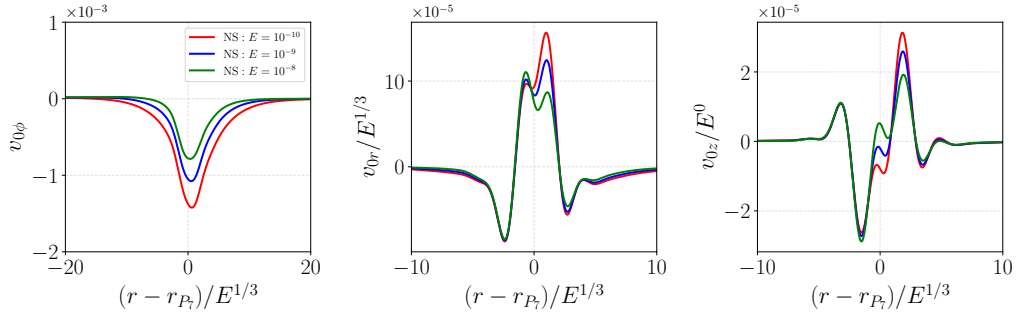

Figure 16: Same caption as figure 12 but for the band originating from P_7 (cut Sr_7).

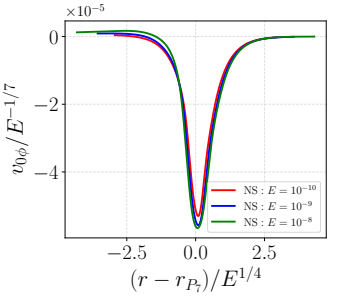

Figure 17: Alternate scaling for the azimuthal velocity in the band originating from P_7 (cut Sr_7).

was already evident in figure 1(e). In figure 18, we confirm that the numerical results align with the theoretically predicted scaling of $E^{1/6}$ for both the azimuthal and axial velocity components.

A faint band originating from the critical point on the outer core is visible in figure 1(a). This band is not associated with the local region P_5, which is not expected to generate any visible band, but rather with P_6. As explained in Appendix C, the structure of the velocity field in this band is particular. Its width scales as $E^{1/6}$, matching with the width of the local region P_6. However, unlike the bands originating from P_3 and P_7, the axial velocity here is significantly weaker than the azimuthal velocity and varies linearly in z. The scaling used in figure 19(a,b) - E^0 for $v_{0\phi}$, $E^{1/2}$ for v_{0z} - are those predicted by the theoretical analysis. The radial velocity, which is expected to be $O(E^{2/3})$ is too weak to be correctly resolved, and has not been plotted. In figure 19(c), we demonstrate that the azimuthal velocity is uniform along the vertical direction, as expected from the theory.

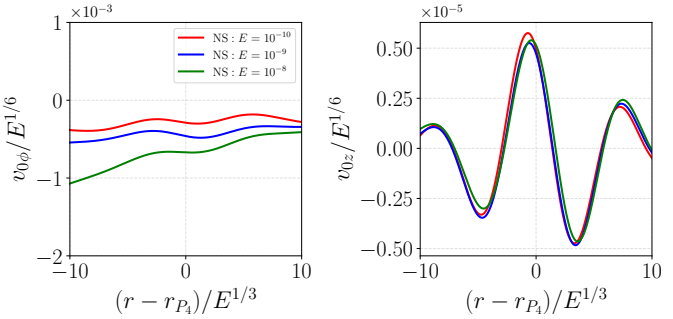

Figure 18: Same caption as figure 12, but for the band originating from P_4 (cut Sr_4). The very weak radial velocity that scales as $E^{1/2}$ has not been plotted.

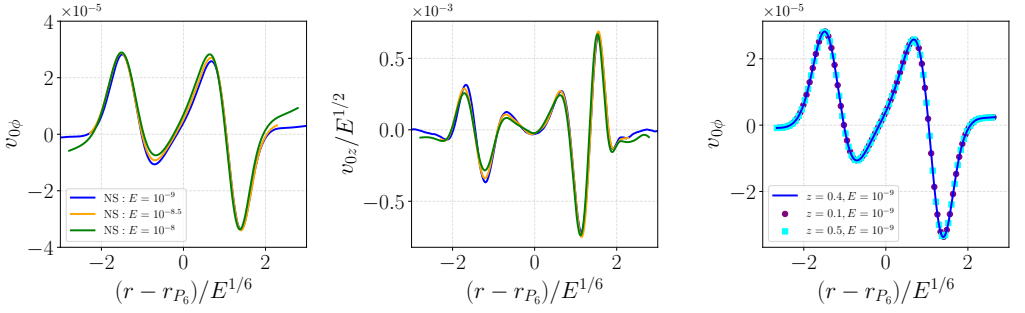

Figure 19: Velocity scalings in the band originating from P_6. (a) Azimuthal velocity versus r along the cut Sr_6; (b) Axial velocity versus r along the cut Sr_6; (c) Azimuthal velocity versus r for different z, along the cut Sr_6, Sr_6^+ and Sr_6^- at $z = 0.4$, $z = 0.5$ and $z = 0.1$, respectively.

6. Conclusion

In this study, we have used both numerical and asymptotic approaches to analyse the meanflow corrections generated by the nonlinear self-interaction of a harmonic solution in a rotating, librating spherical shell. We have focused on the case where the inner core librates at a frequency $\widehat{\omega} = \sqrt{2}\widehat{\Omega}$, for which the harmonic solution exhibits a simple structure, primarily composed of critical point beams propagating along a closed rectangular periodic path. Using the asymptotic structure of the harmonic solution obtained in He *et al.* (2022), we have obtained expressions for the dominant meanflow corrections in the limit of small Ekman numbers. These theoretical predictions have been compared with numerical results obtained for Ekman numbers ranging from 10^{-8} to 10^{-10}, providing strong validation for both the numerical and asymptotic methods. In addition, scaling laws for weaker meanflow bands have been derived and compared against numerical data. This analysis has allowed us to obtain a comprehensive picture of the meanflow correction structure, summarized for the three velocity components in figures 20, 21 and 22.

The main findings are as follows. We have shown that the largest meanflow corrections are localized in the regions where two beams propagating in different directions intersect. In these localized overlap regions, the three velocity components share the same scaling. The largest correction, of order $E^{-1/2}$, occurs in the local region P_1 where the critical beam reaches the rotation axis. Significant corrections of order $E^{-1/6}$ have also been found in regions P_3 and P_7, where the critical beam reflects on the outer boundary, and in region P_4, where it intersects its symmetric counterpart on the equator. Weaker contributions, scaling as E^0 have been observed in regions P_5 and P_6 where the main critical beam interacts with

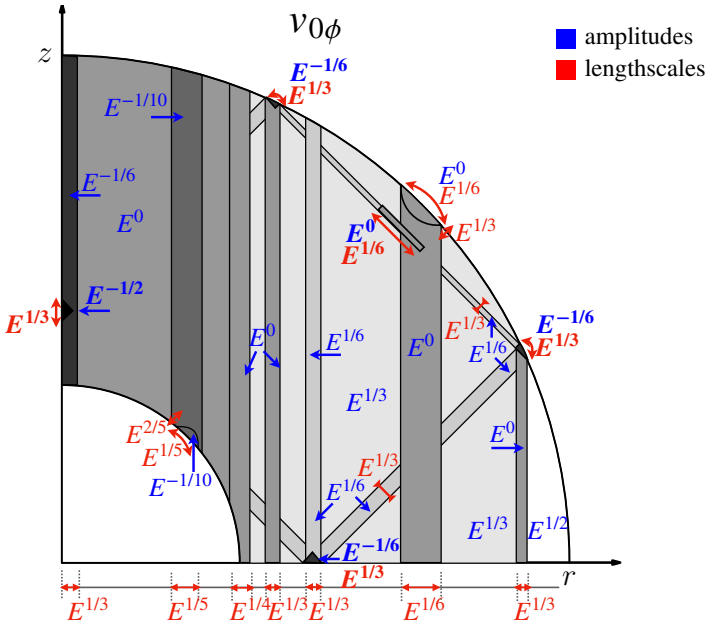

Figure 20: Ekman scaling of the meanflow correction azimuthal velocity $v_{0\phi}$. The blue and red colors denote the amplitude and length scale scalings, respectively. The scalings indicated in bold have been validated by an asymptotic solution.

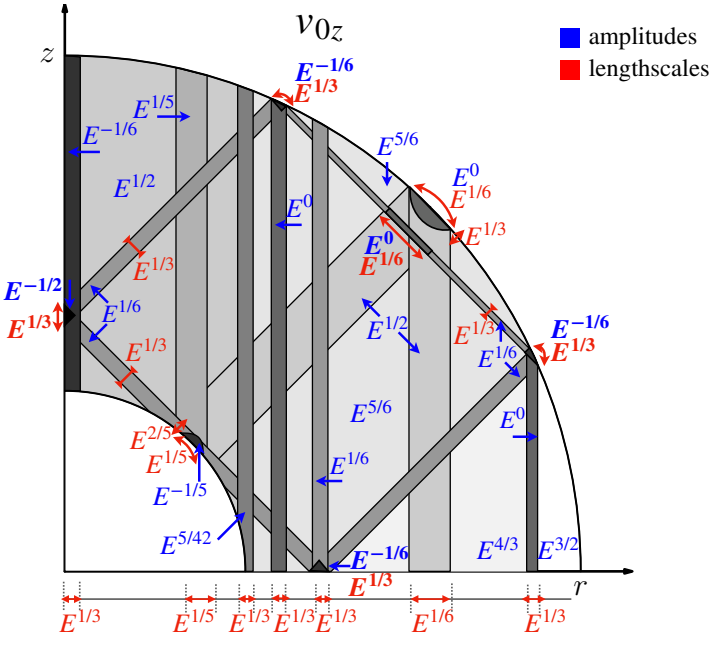

Figure 21: Same caption as in figure 20, but shown for the velocity component v_{0z}.

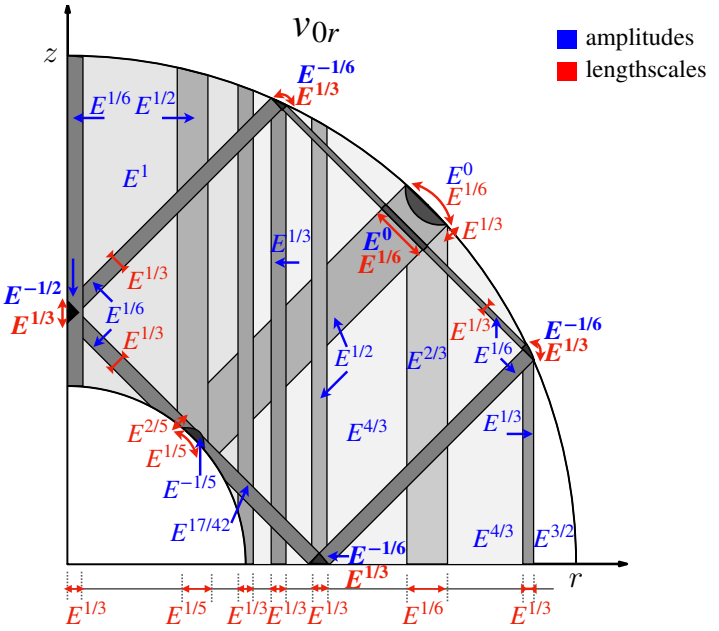

Figure 22: Same caption as in figure 20, but shown for the velocity component v_{0r}.

the wider and weaker secondary beam propagating between the critical points on the inner and outer core.

We have further demonstrated that these strong localized interactions could serve as sources of weaker meanflow bands aligned with the rotation axis. From regions P_3 and P_7 on the outer boundary, we have shown that meanflow bands are generated, characterized by axial and azimuthal velocities of order E^0 and a radial velocity of order $E^{1/3}$, thus confirming the viscous generation mechanism proposed by Le Dizès (2020). When the local interaction occurs away from the boundary, as in regions P_1 and P_4, the resulting meanflow bands are relatively weaker. In all of these bands, of width $E^{1/3}$, the velocity and pressure fields have a complex axial structure and do not conform to classical Taylor-Proudman columns. In contrast, broader meanflow bands are generated from the critical point region P_2 and P_6 located on the inner and outer cores. These bands behave as Taylor-Proudman columns, characterised by a dominant azimuthal velocity component that is independent of the axial coordinate, accompanied by much weaker axial and radial velocity components.

In this study, we have focused on the meanflow correction. However, double-harmonic corrections, scaling as the square of the harmonic solution, are also expected. For the frequency considered, 2ω lies outside the frequency range of inertial waves. As a result, double-harmonic corrections are not expected to propagate. Nevertheless, as explained in Le Dizès (2020), such corrections are still generated within the interaction regions. We expect them to exhibit the same scaling as the meanflow corrections in the local regions P_β, but without the emission of beams from these regions.

In the present study, we have considered a viscous libration forcing of the inner core, which gives rise to an harmonic response of order $E^{1/12}$. For a larger harmonic response, such as that obtained with inviscid forcing, larger meanflow corrections are expected. He *et al.* (2025) considered the same geometry but with a different forcing, corresponding to an inner core vertical oscillation. When the same frequency is used, they showed that a similar harmonic solution is obtained, concentrated along the same rectangular critical ray pattern,

but characterized by a different similarity index, $m = 1/2$, and a larger amplitude scaling as $E^{-1/6}$. A similar analysis can be applied to this solution, which has an amplitude $E^{-1/4}$ larger than the present harmonic solution. We expect analogous results for the meanflow corrections generated by the interaction of the critical beam with itself, specifically within the local interaction regions P_β, as well as in the bands originating from these points. However, since the meanflow corrections scale with the square of the harmonic solution amplitude, all such corrections would be amplified by a factor of $E^{-1/2}$ in that case. In particular, this leads to amplitude scalings of $E^{-3/2}$ in P_1, $E^{-2/3}$ in P_3, P_4 and P_7, and $E^{-1/2}$ in the bands originating from P_3, P_6 and P_7.

Our results give some theoretical grounds to previously observed results in the literature. For example, the fact that differential rotation is preferentially driven at locations where wave beams reflect on the boundaires has been observed in the tidally-driven zonal flows of Favier *et al.* (2014), albeit at much higher Ekman number that those discussed here and for different forcing frequencies. Additionnally, temptative scaling for the volume-averaged energy of the differential rotation were reported in Tilgner (2007) and Favier *et al.* (2014), with exponents ranging from $E^{-1/2}$ to $E^{-3/2}$ (see also the scaling of $E^{-3/10}$ for the azimuthal velocity observed experimentally by Morize *et al.* (2010) but for a full sphere, less relevant to our particular spherical shell). Since these authors considered tidally-dirven flows, their scaling are to be compared with our prediction for an inviscid forcing. From our local analysis, we expect scaling for the volume-averaged energy of the differential rotation of E^{-2} for the local region around P_1 and $E^{-2/3}$ for the local regions around P_3, P_4 and P_7. The volume-averaged energy associated with the bands emanating from P_3, P_6 and P_7 are predicted to scale as $E^{-2/3}$. Our predictions are compatible with existing scaling, in particular with the fact that we expect intense zonal flows in the limit of vanishing Ekman number. Note however that our approach remains valid only in the weakly non-linear limit but clearly disentangles the different zonal flow contributions which was not the case of the volume-averaged approach used in Tilgner (2007) and Favier *et al.* (2014). Note finally that the strongest response on the rotation axis around P_1 might be connected to the focusing effect discussed by Shmakova *et al.* (2021) and Liu *et al.* (2022) in the case of an oscillating torus.

These scalings, for both inviscid and a viscous forcing, define the limits of validity for the weakly nonlinear approach employed in this study. The requirement that the meanflow corrections remain small implies that the forcing amplitude ϵ must be much smaller than $E^{1/4}$ in the viscous case, and much smaller than $E^{3/4}$ in the inviscid case. An even more stringent condition arises from requiring that the meanflow correction remains smaller than the harmonic solution. This leads to the constraint $\epsilon \ll E^{5/12}$ for the viscous case, and $\epsilon \ll E^{7/6}$ in the inviscid case.

It is worth emphasizing that the results presented here are for a harmonic solution with a relatively simple structure, in which the number of beam crossings is limited. For other forcing frequencies, the harmonic response can be significantly more complex, involving multiple reflections of the critical point beams and the possible formation of wave attractors (He *et al.* 2023).

An illustration of such a harmonic response is shown in figure 23(a) for a libration frequency $\widehat{\omega} = 0.8317\widehat{\Omega}$. In figure 23(b), we present the azimuthal velocity contours of the meanflow correction obtained numerically from this harmonic response. Despite the added complexity, many features of the meanflow corrections remain similar. The dominant contribution arising from the boundary layer oscillation of the inner core is still present. Localized contributions at beam intersection points in the bulk are also clearly visible, as are bands originating from locations where beams reflect on the boundaries. For both the localized contributions and the emitted bands, a similar asymptotic analysis is expected to hold. In fact, the structure

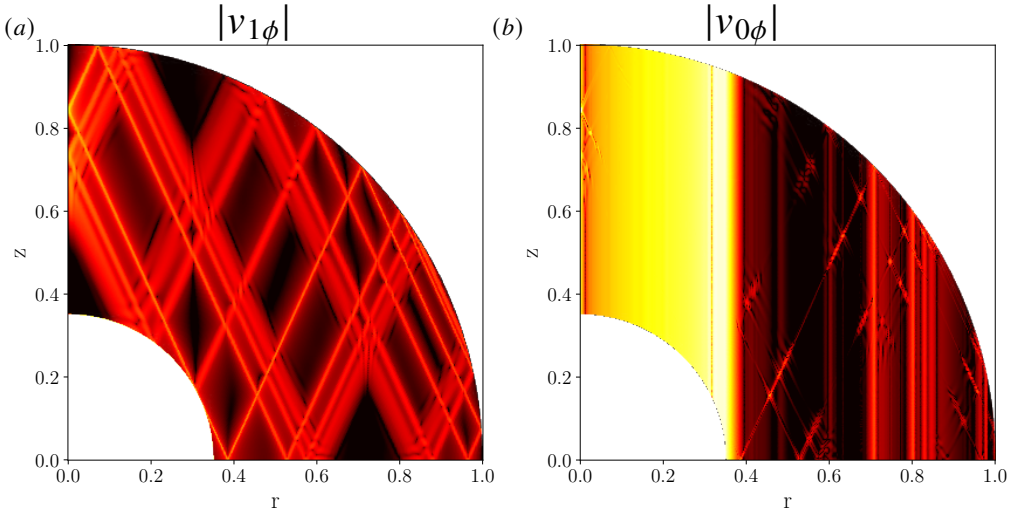

Figure 23: (a) Contours of amplitudes of harmonic linear azimuthal velocity $|v_{1\phi}|$. The blue lines are a critical line from the source point S_c. (b) Meanflow azimuthal velocity $|v_{0\phi}|$ at $E = 10^{-9}$ and the libration frequency of the inner core is $\omega = 0.8317$.

of the solution may be simpler in this case, as the main beam is now a single critical point beam (with the same similarity index $m = 5/4$ and amplitude scaling as $E^{1/12}$), propagating in only one direction - unlike in our previous case, which involved an infinite sum of beams propagating in both directions. As long as the analysis is restricted to regions away from the attractor, we therefore expect similar localized contributions of order $E^{-1/6}$ at beam intersections, and bands of width $O(E^{1/3})$, with velocity scaling as E^0, originating from reflection points on the outer boundary.

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Declaration of interests. The authors report no conflict of interest.

Supplementary materials. The directory stores the data, scaling parameters, and Jupyter notebook used for the validation of Ekman scaling at various cut positions (shown in figure 4) from numerical simulations at https://cocalc.com/share/public_paths/bd308dadc457d695975b37acd237ba8e89f75ded/Figure_num_scaling. This validation corresponds to figure 6 and figures in section 5.

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Appendix A. Numerical convergence details

The pseudo-spectral method workflow is shown in figure 24. Based on the linear velocity in spectral space $\check{v}_1$ of the harmonic solution, this paper calculates the steady Reynolds stress through the pseudo-spectral method and then obtains the meanflow. For the angular

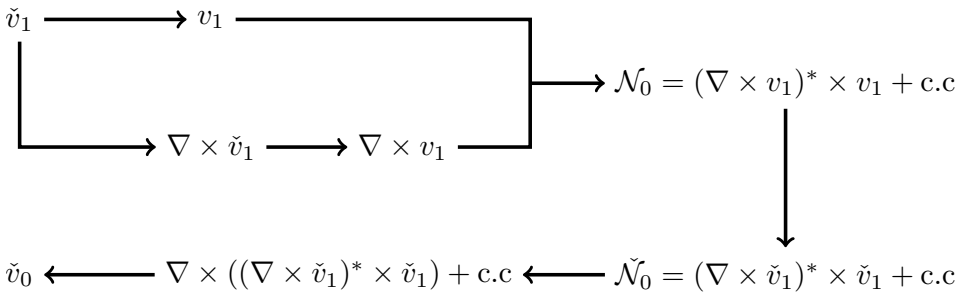

Figure 24: Schematic of the pseudo-spectral method workflow.

Ekman number	Chebyshev grid resolution N	Spherical harmonic grid resolution L
$E = 10^{-6}$	300	600
$E = 10^{-7}$	300	900
$E = 10^{-8}$	900	3000
$E = 10^{-9}$	900	3500
$E = 10^{-10}$	2500	8000

Table 1: Resolutions of different Ekman numbers

components, we employ the SHTns package to efficiently transform spherical harmonic coefficients to physical space (Schaeffer 2013).

We tested two dealiasing approaches: traditional 2/3 truncation for the spherical harmonic coefficients and the SHTns package's anti-aliasing capability for angular dimensions (Schaeffer 2013). No dealiasing was needed in the radial direction as we used Chebyshev differentiation matrices in physical space. Since both methods yielded identical results to simulations without dealiasing at our resolution, we present all results without applying dealiasing operations.

The convergence of the spectral codes with various resolutions is tested by the spectra of the Chebyshev coefficients and the spherical harmonic components, as in Rieutord & Valdettaro (1997). Figure 25 shows the spectra at the smallest Ekman number $E = 10^{-10}$.

We have verified that for all Ekman numbers, the resolutions shown in table 1 ensure that the tolerance $\mathrm{Es}_{\min}/\mathrm{Es}_{\max}$ of the energy spectrum is less than 10^{-8}.

Appendix B. Parameters for the harmonic solution

The parameters defining the harmonic solution along the main rectangular circuit are given in tables 2 and 3 for the northward and southward beam, respectively. They use the following quantities

$$L = l_1 + l_2 + l_3 K^{-3} + l_4 + l_5 \tag{B 1}$$

with

$$l_1 = \eta, \quad l_2 = \sqrt{1-\eta^2} - \eta, \quad l_2 = 2\eta, \quad l_4 = l_2, \quad l_5 = l_1, \tag{B 2}$$

and

$$K = \frac{\sin(\alpha + \pi/4)}{\sin(\alpha - \pi/4)}. \tag{B 3}$$

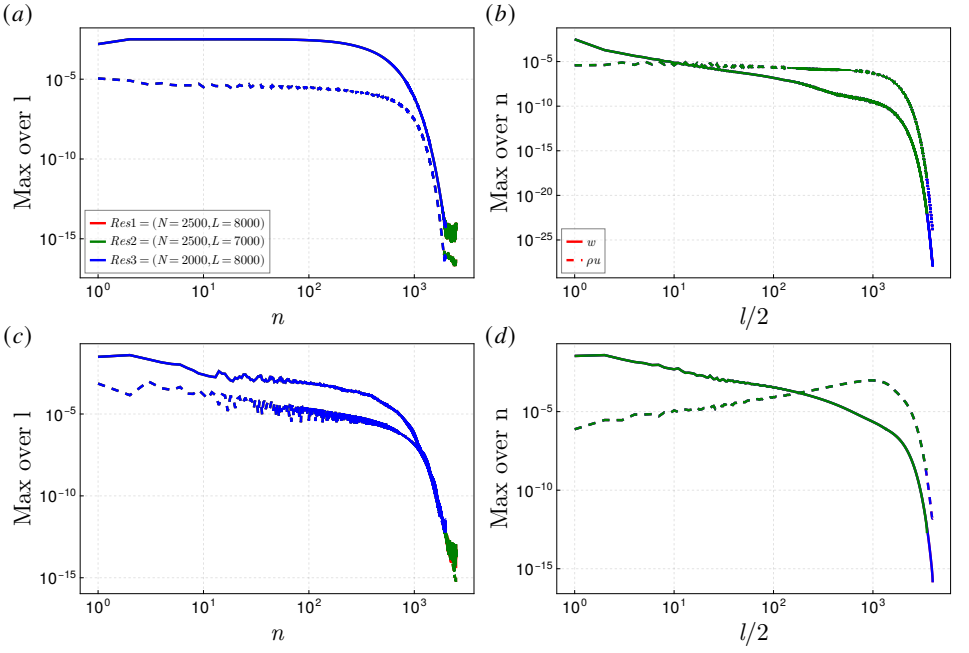

Figure 25: Spectra compare of (a) linear Chebyshev coefficients, (b) linear spherical harmonic components, (c) nonlinear Chebyshev coefficients, and (d) nonlinear spherical harmonic components at $E = 10^{-10}$ with three different resolutions $Res1 = (2500, 8000)$, $Res2 = (2500, 7000)$, and $Res3 = (2000, 8000)$, red, green and blue lines respectively. The toroidal component (w solid line) and the radial component (ρu dashed line) have been separated. For each l and n, the maximum value over the other spectral component is chosen.

Critical line	$x_{\parallel}^N$	$x_{\perp}^N$	$\tilde{C}_0^N$	L^N
$S_c \to P_1$	$\dfrac{-r+z}{\sqrt{2}}$	$\dfrac{r+z-\sqrt{2}\eta}{\sqrt{2}}$	$\tilde{C}_0^N$	L
$P_1 \to P_3$	$l_1 + \dfrac{r+z-\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{-r+z-\sqrt{2}\eta}{\sqrt{2}}$	$\mathrm{i}\tilde{C}_0^N$	L
$P_3 \to P_7$	$(l_1+l_2)K^3 + \dfrac{r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{-r-z+\sqrt{2-2\eta^2}}{\sqrt{2}}$	$\mathrm{i}K^{-1/4}\tilde{C}_0^N$	$K^3 L$
$P_7 \to P_4$	$l_1+l_2+l_3K^{-3} + \dfrac{-r-z+\sqrt{2-2\eta^2}}{\sqrt{2}}$	$\dfrac{r-z-\sqrt{2}\eta}{\sqrt{2}}$	$\mathrm{i}\tilde{C}_0^N$	L
$P_4 \to S_c$	$l_1+l_2+l_3K^{-3}+l_4 + \dfrac{-r+z+\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{r+z-\sqrt{2}\eta}{\sqrt{2}}$	$\mathrm{i}\tilde{C}_0^N$	L

Table 2: Northward path coordinates, amplitudes and propagation distances within one cycle.

Appendix C. Meanflow bands

In this section, we analyse the structure of the meanflow bands that are created within the bulk from the local interaction regions P_β. These bands have the particularity to be present in fluid regions where the Reynolds stress is very small. The corresponding velocity field is therefore expected to satisfy homogeneous equations.

Critical line	$x_\parallel^S$	$x_\perp^S$	$\tilde{C}_0^S$	L^S
$S_c \to P_4$	$\dfrac{-r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{-r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\tilde{C}_0^S$	L
$P_4 \to P_7$	$l_5+\dfrac{r+z-\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{-r+z+\sqrt{2}\eta}{\sqrt{2}}$	$\tilde{C}_0^S$	L
$P_7 \to P_3$	$(l_5+l_4)K^3+\dfrac{-r+z+\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{r+z-\sqrt{2-2\eta^2}}{\sqrt{2}}$	$K^{-1/4}\tilde{C}_0^S$	K^3L
$P_3 \to P_1$	$l_5+l_4+l_3K^{-3}+\dfrac{-r-z+\sqrt{2-2\eta^2}}{\sqrt{2}}$	$\dfrac{r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\tilde{C}_0^S$	L
$P_1 \to S_c$	$l_5+l_4+l_3K^{-3}+l_2+\dfrac{r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\dfrac{-r-z+\sqrt{2}\eta}{\sqrt{2}}$	$\mathrm{i}\tilde{C}_0^S$	L

Table 3: Southward path coordinates, amplitudes and propagation distances within one cycle.

C.1. *Bands issued from P_3, P_4 and P_7*

We first consider bands generated from the local regions P_3, P_4 and P_7. These regions have a width of order $E^{1/3}$ and we can expect the bands to have a same width. It is thus natural to introduce the local radial variable $\tilde{r} = E^{-1/3}(r - r_{P_\beta})$ and to use the following ansatz for the velocity field:

$$\mathbf{v}_0 = (E^{1/3}v_{0r}, v_{0\phi}, v_{0z}, E^{1/3}p_0)(\tilde{r}, z). \tag{C 1}$$

The velocity field then satisfies the following set of equations

$$-2v_{0\phi} + \frac{\partial p_0}{\partial \tilde{r}} = 0, \tag{C 2a}$$

$$2v_{0r} - \frac{\partial^2 v_{0\phi}}{\partial \tilde{r}^2} = 0, \tag{C 2b}$$

$$\frac{\partial p_0}{\partial z} - \frac{\partial^2 v_{0z}}{\partial \tilde{r}^2} = 0, \tag{C 2c}$$

$$\frac{\partial v_{0r}}{\partial \tilde{r}} + \frac{\partial v_{0z}}{\partial z} = 0. \tag{C 2d}$$

For P_3 and P_7, the solution should be valid from $-z_{P_\beta}$ to z_{P_β} with a condition of anti-symmetry on the axial velocity component with respect to the equatorial plane. This implies that the general solution for the bands issued from P_3 and P_7 takes the form

$$v_{0r} = -\frac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} A(k)k^3\cosh(k^3z/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 3a}$$

$$v_{0\phi} = -\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} A(k)k\cosh(k^3z/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 3b}$$

$$v_{0z} = -\frac{1}{2}\int_{-\infty}^{+\infty} A(k)k\sinh(k^3z/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 3c}$$

$$p_0 = \int_{-\infty}^{+\infty} A(k)\cosh(k^3z/2)e^{\mathrm{i}k\tilde{r}}dk. \tag{C 3d}$$

where $A(k)$ is a function determined by the boundary condition at $z = z_{P_\beta}$. In particular, $A(k)$ is set by the Ekman pumping generated at the boundary close to P_β. We have seen that, at the order $E^{-1/6}$, the Ekman pumping vanishes. This explains why the order of the band is smaller than $E^{-1/6}$. As shown in Le Dizès (2020), the Ekman pumping first appears at the order E^0. If we denote by $w_\infty(\tilde{r}) = \tilde{v}_{0z}(\tilde{r}, \tilde{z} = -\infty)$ the Ekman puming at this order, the function $A(k)$ is just obtained by the condition $v_{0z}(\tilde{r}, z_{P_\beta}) = w_\infty(\tilde{r})$, which leads to

$$A(k) = -\frac{2\hat{w}_\infty}{k \sinh(k^3 z_{P_\beta}/2)}\,, \tag{C 4}$$

where $\hat{w}_\infty$ is the Fourier transform of w_∞.

For the band issued from the point P_4 on the equatorial plane, the solution must satisfy non-penetration conditions at the ends of the band, corresponding to $z = \pm z_5 = \pm\sqrt{1-2\eta^2}$. This leads to the following form for $z > 0$:

$$v_{0r} = -\frac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} A(k)k^3 \cosh(k^3(z-z_5)/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 5a}$$

$$v_{0\phi} = -\frac{\mathrm{i}}{2}\int_{-\infty}^{+\infty} A(k)k \cosh(k^3(z-z_5)/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 5b}$$

$$v_{0z} = -\frac{1}{2}\int_{-\infty}^{+\infty} A(k)k \sinh(k^3(z-z_5)/2)e^{\mathrm{i}k\tilde{r}}dk, \tag{C 5c}$$

$$p_0 = \int_{-\infty}^{+\infty} A(k) \cosh(k^3(z-z_5)/2)e^{\mathrm{i}k\tilde{r}}dk. \tag{C 5d}$$

The amplitude $A(k)$ is now prescribed by the axial flux generated from the Reynolds stress close to $z = 0$ around P_4. We have seen that it is zero at the order $E^{-1/6}$. We further claim that it also vanishes at the orders $E^{-1/12}$ and E^0, due to the similar form of the Reynolds stress at these orders. Indeed, as shown in section 3.5, up to $O(E^{1/3})$, the viscous corrections to the leading-order harmonic solution share the structure as the main beam. They also satisfy the two key properties -(3.10) and (3.11)- that were used to derive the meanflow correction expression (4.4), which notably exhibits no axial flux.

The first correction that generates a non-zero axial flux from P_4 is therefore expected to appear at a magnitude $E^{1/3}$ smaller than the dominant term - that is, at the order $E^{1/6}$. If this flux is denoted as $E^{1/6}w_\infty(\tilde{r})$, the function $A(k)$ is obtained by enforcing the condition $v_{0z}(\tilde{r}, z = 0) = E^{1/6}w_\infty(\tilde{r})$, leading to

$$A(k) = E^{1/6}\frac{2\hat{w}_\infty}{k \sinh(k^3 z_{Q_5}/2)}\,. \tag{C 6}$$

We therefore expect the band generated from P_4 to be of order $E^{1/6}$ smaller than those originating from P_3 and P_7.

It is worth emphasizing that the velocity field in the bands issued from the three points P_3, P_4 and P_7 depends on the axial coordinate z. These bands are therefore not Taylor-Proudman columns, in which axial and azimuthal velocity components are invariant along the rotation axis. Here, due the smallness of the radial scale, the viscous effects responsible for axial variations become observable over the O(1) axial extent of the column.

C.2. *Band issued from P_1*

For the band originating from the point P_1, the analysis differs slightly due to the presence of the cylindrical singularity. We still introduce the local variable $\tilde{r} = E^{-1/3}r$, and adopt the

same ansatz (C 1) for the meanflow corrections. The governing equations become:

$$-2v_{0\phi} + \frac{\partial p_0}{\partial \tilde{r}} = 0, \tag{C 7a}$$

$$2v_{0r} - \left(\frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2}\right) v_{0\phi} = 0, \tag{C 7b}$$

$$\frac{\partial p_0}{\partial z} - \left(\frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\right) v_{0z} = 0, \tag{C 7c}$$

$$\frac{\partial v_{0r}}{\partial \tilde{r}} + \frac{v_{0r}}{\tilde{r}} + \frac{\partial v_{0z}}{\partial z} = 0. \tag{C 7d}$$

On either side of P_1, we obtain solutions that satisfy the non-penetration condition at $z = z_1^+ = 1$ and at $z = z_1^- = \eta$, respectively. These solutions take the following form:

$$v_{0r}^{\pm} = \frac{1}{4}\int_{-\infty}^{+\infty} A^{\pm}(k)k^3 \cosh(k^3(z - z_1^{\pm})/2)J_1(k\tilde{r})dk, \tag{C 8a}$$

$$v_{0\phi}^{\pm} = -\frac{1}{2}\int_{-\infty}^{+\infty} A^{\pm}(k)k \cosh(k^3(z - z_1^{\pm})/2)J_1(k\tilde{r})dk, \tag{C 8b}$$

$$v_{0z}^{\pm} = -\frac{1}{2}\int_{-\infty}^{+\infty} A^{\pm}(k)k \sinh(k^3(z - z_1^{\pm})/2)J_0(k\tilde{r})dk, \tag{C 8c}$$

$$p_0^{\pm} = \int_{-\infty}^{+\infty} A^{\pm}(k) \cosh(k^3(z - z_1^{\pm})/2)J_0(k\tilde{r})dk, \tag{C 8d}$$

where the superscript + refers to the region $z_{P_1} < z \leqslant 1$, and the superscript − refers to $\eta \leqslant z < z_{P_1}$. The two functions $A^{\pm}(k)$ are obtained by the conditions of matching with the local solution close to P_1. As previously shown, at the order $E^{-1/2}$, the local solution is confined to the vicinity of P_1, and thus does not generate any meanflow bands. For the same reasons as in the case of P_4, the corrections at the orders $E^{-5/12}$ and $E^{-1/3}$ are also localized and do not induce any non-local band structure. The first non-localized contribution is expected to arise at the order $E^{-1/6}$. The functions $A^{\pm}$ can be related to the jumps $E^{-1/6}\delta\hat{v}$ and $E^{-1/6}\delta\hat{w}$ of the azimuthal and axial velocity across the local region around P_1 by the following relations

$$v_{0\phi}^{+}(\tilde{r}, z = z_{P_1}) - v_{0\phi}^{-}(\tilde{r}, z = z_{P_1}) = E^{-1/6}\delta\hat{v}(\tilde{r}), \tag{C 9a}$$

$$v_{0z}^{+}(\tilde{r}, z = z_{P_1}) - v_{0z}^{-}(\tilde{r}, z = z_{P_1}) = E^{-1/6}\delta\hat{w}(\tilde{r}). \tag{C 9b}$$

As with the bands originating from P_3, P_4 and P_7, the band emanating from P_1 also exhibit axial dependence.

C.3. *Bands issued from P_2 and P_6*

The bands issued from the local regions P_2 and P_6 differ from those previously discussed, as their radial width is larger than $E^{1/3}$. As a consequence, the azimuthal velocity can no longer depend on the axial coordinate z. These bands are then Taylor-Proudman columns.

For P_6, the appropriate radial variable and velocity ansatz are $\check{r} = (r - r_{P_6})/E^{1/6}$ and

$$\mathbf{v}_0 = (E^{2/3}v_{0r}, v_{0\phi}, E^{1/2}v_{0z}, E^{1/6}p_0)(\check{r}, z). \tag{C 10}$$

This leads to the following governing equations:

$$-2v_{0\phi} + \frac{\partial p_0}{\partial \check{r}} = 0, \tag{C 11a}$$

$$2v_{0r} - \frac{\partial^2 v_{0\phi}}{\partial \check{r}^2} = 0, \tag{C 11b}$$

$$\frac{\partial p_0}{\partial z} = 0, \tag{C 11c}$$

$$\frac{\partial v_{0r}}{\partial \check{r}} + \frac{\partial v_{0z}}{\partial z} = 0. \tag{C 11d}$$

This ansatz and these equations above show that an azimuthal velocity of order 1 in the bulk can only be compatible with an axial velocity of order $E^{1/2}$. In particular, (C 11b,d) imply that the axial velocity, which has to be antisymmetric with respect to the equatorial plane, must satisfy

$$v_{0z} = -z\frac{\partial v_{0r}}{\partial \check{r}} = -\frac{z}{2}\frac{\partial^3 v_{0\phi}}{\partial \check{r}^3}, \tag{C 12}$$

for a given azimuthal velocity profile $v_{0\phi}(\check{r})$. However, the $O(1)$ particular solution that we computed in §4.2.2 is expected to induce Ekman pumping of order $E^{1/3}$. Similarly, an $O(1)$ azimuthal velocity in the bulk also generates Ekman pumping of the same order. These two contributions must cancel each other to ensure compatibility with the axial flow in the bulk. This matching condition prescribes the function $v_{0\phi}(\check{r})$, from which the full velocity and pressure field in the band is entierely determined.

The case of P_2 is more complex, as both $E^{1/5}$ and $E^{1/6}$ radial scales are, a priori, possible. Assuming the $E^{1/5}$ scale dominates, we introduce $\bar{r} = (r - r_{P_2})/E^{1/5}$ and the ansatz

$$\mathbf{v}_0 = (E^{3/5}v_{0r}, v_{0\phi}, E^{2/5}v_{0z}, E^{1/5}p_0)(\bar{r}, z). \tag{C 13}$$

The resulting equations are analogous to those in (C 11) with $\check{r}$ replaced by $\bar{r}$. As in the case of P_6, the leading-order azimuthal velocity component must be independent of z. However, the axial velocity may now either be linear in z with the scaling prescribed by the ansatz (C 13), or independent of z and larger in magnitude. It is important to note that the bulk solution with an $O(1)$ azimuthal velocity is expected to generate $O(E^{3/10})$ Ekman pumping at both the inner and outer cores. These Ekman pumpings differ on each boundary and are larger than the possible linear axial flow in the bulk. Therefore, the azimuthal velocity in bulk must be chosen such that the Ekman pumping generated in P_2, by the solution forced by the Reynolds stress, exactly cancels the Ekman pumping produced by the bulk solution at the outer boundary. This cancellation allows for a z-independent axial flow in the bulk at leading order. Although evaluating the Ekman pumping precisely is challenging, the above reasoning enables us to deduce the relative orders of the velocity components. For instance, if the Ekman pumping in P_2 is $O(E^{1/5})$, then, in the bulk, we expect: $v_{0\phi} = O(E^{-1/10})$, $v_{0z} = O(E^{1/5})$, and $v_{0r} = O(E^{1/2})$. A weak axial dependence of the axial velocity is also expected at the order $E^{3/10}$ and given by the analogue of equation (C 12) for this band.

To conclude this section, we briefly comment on the band originating from P_5. As in the case of P_1, such a band is generated by the jumps of $v_{0\phi}$ and v_{0z} across the local region P_5. Due to the different scales of the two interacting beams, these jumps are expected to be of order $E^{2/3}$. The jump of v_{0z} must be compensated by an Ekman pumping of the same order at the outer boundary. This requires an azimuthal velocity in the band of order $E^{1/3}$. This velocity amplitude is comparable to the background azimuthal velocity in the bulk, which explains why no distinct band originating from P_5 is visible in figures 1 and 5(b).

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