Internal wave beam propagation in nonuniform stratifications

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Talk Overview

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Internal Waves – Introduction

- **Internal waves** – Propagating disturbances of the density stratification $\rho_0(z)$ of a stably stratified fluid

- Small two-dimensional perturbations in an incompressible, inviscid, non-rotating, Boussinesq fluid are governed by:

\[
\frac{\partial^2}{\partial t^2} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right] w + N^2 \frac{\partial^2 w}{\partial x^2} = 0
\]

$w$ – vertical velocity perturbation

\[
N^2 = -\frac{g}{\rho_0} \frac{d\rho_0}{dz}
\] - Stratification (Brunt-Vaisala) frequency

- characteristic of the stable density profile only
- Traveling plane wave solutions for constant $N$

$$w = W e^{i k(x - z \cot \theta) - i \omega t} + c.c.$$  

where

$$\sin \theta = \frac{\omega}{N}$$

$0 < \omega < N$ - propagating waves

$\omega > N$ - evanescent (in the vertical direction) waves

- Direction of energy propagation given by:
Motivation

Internal waves in the ocean and the atmosphere

Monterey Bay, California.
(Lien & Gregg 2001)

Finite-width internal wave beams occur in geophysical settings

(Fovell et. al. 1992)
Motivation
Stratification profiles in the ocean and the atmosphere

Ocean

Atmosphere

Bay of Biscay
(Lam et. al. 2004)

(Walterscheid et. al. 2004)

Nonuniformities ubiquitous in geophysical settings
Plane wave transmission – $N_1$ to $N_2$

Continuity of pressure and velocity at the interface determines all transmission and reflection properties.

\[
T_c = \frac{4 \cot \theta_1 \cot \theta_2}{(\cot \theta_1 + \cot \theta_2)^2},
\]

\[
T_w = \frac{2 \cot \theta_1}{\cot \theta_1 + \cot \theta_2},
\]

\[
T_{\Delta N^2} = \frac{2 \cot \theta_2}{\cot \theta_1 + \cot \theta_2} \frac{\sin^2 \theta_1}{\sin^2 \theta_2}
\]
Plane wave transmission – $N_1$ to $N_2$

\[
N^2 = \frac{N_2^2 - N_1^2}{2} \tanh\left(\frac{z}{L}\right) + \frac{N_1^2 + N_2^2}{2}
\]

\[\psi\] - streamfunction

\[
\psi_I = \Psi_I e^{[ik(x-z \cot \theta_1) - i\omega t]} + c.c.
\]

\[
\psi_R = \Psi_R e^{[ik(x+z \cot \theta_1) - i\omega t]} + c.c.
\]

\[
\psi_T = \Psi_T e^{[ik(x-z \cot \theta_2) - i\omega t]} + c.c.
\]

\[
\sin \theta_1 = \omega/N_1
\]

\[
\sin \theta_2 = \omega/N_2
\]

Transmission and reflection coefficients calculated by solving (Nault and Sutherland 2007):

\[
\phi'' + k^2 \left( \frac{N^2}{\omega^2} - 1 \right) \phi = 0
\]

where \(\psi = \Re \left( \phi(z) e^{[i(kx - \omega t)]} \right)\)
Plane wave transmission – $N_1$ to $N_2$

$T_e$ for $Lk = 0.1$

$T_e$ for $Lk = 1$

$Lk_{\text{min}}(\cot \theta_1, \cot \theta_2) \gg 1$, WKB-like, perfect transmission

$Lk_{\text{max}}(\cot \theta_1, \cot \theta_2) \ll 1$, sharp-interface-like
Internal Wave Beams

- Finite-width wave beams are composed of contributions from individual plane waves

- Upward propagating left-to-right individual plane waves written as:

  \[ w = \text{Re}[W \exp(ik(x - \cot \theta z))e^{-i\omega t}] \]

- Upward propagating left-to-right finite-width wavebeams written as:

  \[ w = \text{Re}[e^{-i\omega t} \int_{-\infty}^{\infty} W(k)\exp(ik(x - \cot \theta z))dk] \]

Transmission problem solved for every wavenumber present in the wave beam.
Wave beam filtering by a finite-width interface

Can the wave beam become unstable upon transmission? – Linear theory says yes, but the linear theory is not reliable in such a scenario.
Including viscosity

\[ \phi'' + k^2 \left( \frac{N^2}{\omega^2} - 1 \right) \phi = \frac{i \nu}{\omega} \left( \phi'''' - 2k^2 \phi'' + k^4 \phi \right) \]

where \( \psi = \Re \left( \phi(z) e^{i(kx-\omega t)} \right) \)

We now have a fourth order equation to solve for the transmission and reflection properties.
A simple model for the upper ocean

$T_e$ symmetric w.r.t. interchange in $N_1$ and $N_2$, implies all the energy ultimately comes back out. But, how does it all come back out?
Laboratory Experiments - Details

Acrylic tank – 1.28m long, 0.66m tall, 0.2m thick.

Random pattern of dots at 1.25 m behind and camera at 3.18m in front of the tank for synthetic schlieren measurements.

Nonlinear salt water stratifications.

An internal wave generator, based on the design given by Gostiaux et. al. (2007), produces unidirectional wave beams.
Laboratory Experiment – $L^*/L_h \sim 1$

$N_1 = 0.89 \text{ rad/s}$

$N_{\text{max}} = 1.44 \text{ rad/s}$

$N_2 = 1.34 \text{ rad/s}$

$L^*/L_h \sim 1$

Broadly scattered reflected wave field.

Detectable signal as far away as $x = 3L_h$.

Excellent agreement between experiment and theory.
Laboratory Experiment – $L^*/L_h > 1$

Reflected wave field contains distinct wave beams. Wave beam from the second interaction site much stronger than the first.

Wave beam ducting – significant activity in the $N_2$ layer at $x = 0.4m$.

$N_1 = 0.89 \text{ rad/s}$

$N_{\text{max}} = 1.94 \text{ rad/s}$

$N_2 = 1.62 \text{ rad/s}$

$L^*/L_h > 2$
A relevant ocean observation

Internal wave measurements near the Keana ridge, Hawaii reported in Martin et. al. 2005.

(a) The mean kinetic energy density reveals the existence of an internal wave beam generated at the Keana ridge. (b,c) A time-averaged and an individual density profile from the Keana ridge at the time of the data in (a).

1. No obvious reflected wave beam from the ocean surface

2. Periodic features in the vertical direction
Can nonuniform N account for the observed features?

1. Time-averaged profile reflects the incident wave beam almost completely from the ocean surface.

2. Small-scale features in the individual profile can cause wave beam ducting in the upper layer and produces vertical periodicity.
Conclusions

Nonuniform stratifications can cause significant amplification/diminution of various physical quantities.

Wave beams are scattered by changing stratifications, providing insights into the behavior of wave beams near the upper ocean.

Some predictions by linear theory experimentally verified, scope for more experiments.

Nonlinear effects in the upper ocean? – Nicolas will take over.
Thank You