Spontaneous imbalance and gravity wave radiation from vortical flow in a rotating shallow water system

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1. Motivation
   - Limitation of Balanced Flow
   - Spontaneous Gravity Wave (GW) radiation
   - Spectral like Combined Compact Difference scheme

2. Experimental setup
   - Basic equation & Basic state
   - Conditions of numerical simulation

3. Results
   - Continuous GW radiation
   - Effect of the earth rotation
   - GW source and its analysis (Lighthill theory)

4. Summary
1. Motivation

Stratified and rotating flows in large scales are nearly balanced... But there is a limitation. Lorenz(1980), Leith(1980)  Slow manifold & super balance
Ford et al.(2000)  Slow quasimanifold
Sugimoto et al. (2007a)  Breakdown of balance

\[ Ro=100, \quad Fr=0.5 \]

Breakdown of balance due to GW radiation
GW play a significant role on the middle atmosphere

GW radiation, propagation, and dissipation

PANSY project (2005)
Spontaneous GW radiation (GW radiation from vortical flow)

- **Observational study**
  - Yoshiki and Sato (2000): polar night jet
  - Kitamura and Hirota (1989): sub tropical jet
  - Pfister *et al.* (1993): hurricane

- **Experimental study**
  - Williams *et al.* (2005): 2-layer fluid in rotating annuls

- **Numerical study** (GCM, Meso-scale model)
  - O’Sullivan and Dunkerton (1995): sub tropical jet
  - Zhang (2004): sub tropical jet
  - Sato *et al.* (2005): polar night jet
  - Plougonven and Snyder (2005): sub tropical jet

(Dalin *et al.* 2004)

Williams *et al.* (2005)

Plougonven *et al.* (2005)
Numerical study
(simplified model = f-plane Shallow Water, SH)

Ford (1994): vorticity stripe
Sugimoto et al. (2005, 2007b, 2008etc):
unsteady jet with relaxation forcing

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv &= -g \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu &= -g \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} + (H + \eta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0.
\end{align*}
\]

• The most simplified system in which rotational flows and GW exist.
• Same as 2D compressible gas fluid if the earth rotation is negligible.
• GW are analogous to Sound Waves. **Lighthill theory**
Analogy with vortex sound (Lighthill theory)

Gravity wave source (Lighthill-Ford eq.)

\[
\left( \frac{\partial^2}{\partial t^2} + f^2 - gH_0 \Delta \right) \frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} \left( \frac{\partial (hu^2)}{\partial t} + fhuv + \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \right) \\
\text{eq. (1)} \\
+ \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial (2huv)}{\partial t} + fhv^2 - fhu^2 \right) \\
+ \frac{\partial^2}{\partial y^2} \left( \frac{\partial (hu^2)}{\partial t} - fhuv + \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \right) \\
\text{eq. (2)} \\
+ \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial (2huv)}{\partial t} + fhv^2 - fhu^2 \right) \\
+ \frac{\partial^2}{\partial y^2} \left( \frac{\partial (hu^2)}{\partial t} - fhuv + \frac{g}{2} \frac{\partial}{\partial t} (h - H_0)^2 \right) \\
\text{eq. (3)}
\]

GW amplitude for far field

\[
\frac{\partial \bar{h}(y, t)}{\partial t} = \frac{1}{2c_0} \int_0^t dt' \bar{G}(y + c_0 (t - t'), t') - \bar{G}(y - c_0 (t - t'), t'),
\]

\( \bar{G}(y, t) \): Zonally averaged source

Power law of GW flux

\[
F_e \sim \frac{\omega B^2 U_0^3 \left( \omega^2 - f^2 \right)^{\frac{1}{2}}}{Fr} \frac{1}{g}
\]

Corresponds to the power law of Mach number if \( f = 0 \)
Parameter dependence of gravity wave flux \( (Fr, Ro) \)

- **Fr dependence at** \( Ro=10 \)
- **Ro dependence at** \( Fr=0.1 \)

Breakdown of the power law of \( Fr \)

Local maximum at \( Ro=10 \)

- **Frequency spectra of GW source** (\( Ro \) dependence at \( Fr=0.1 \))

\[
T_{22} = \frac{\delta (hu^2)}{\partial t} + \frac{\delta hu}{\partial t} - \frac{g \partial h^2}{2 \partial t}
\]

- \( Ro=100 \)
- \( Ro=1 \)

Sugimoto et al. (2008)

Inertial frequency cut-off at smaller \( Ro \)

Increase of the source related to Coriolis term at smaller \( Ro \)

Cut-off \( (f/2\pi) \)
This study

We investigate spontaneous GW radiation in SH on a rotating sphere with high accuracy numerical model.

- To focus on the latitudinal change of the earth rotation

Energy of gravity waves $<<$ Energy of rotational flows

- We need a special numerical model with high resolution

Spectral like Combined Compact Difference scheme (Sp-CCD)
Numerical schemes

- **Spectral scheme**
  - high accuracy • high resolution
  - complicated • large memory • slow
  -> not usable for parallel scalar machine

- **Finite difference scheme with high accuracy**
  Compact Difference scheme, CD
  Combined CD scheme, CCD (Chu & Fan, 1998)
  -> achieves high accuracy with few stencils

\[
\begin{align*}
  f_i' &= a_1(f_{i+1}' + f_{i-1}') + b_1 h (f_{i+1}'' - f_{i-1}'') + c_1 h^2 (f_{i+1}''' + f_{i-1}''') = \frac{d_1}{h} (f_{i+1} - f_{i-1}) \\
  f_i'' &= a_2 \frac{h}{h} (f_{i+1} - f_{i-1}) + b_2 (f_{i+1}'' + f_{i+1}'') + c_2 h (f_{i+1}''' - f_{i-1}''') = \frac{d_2}{h^2} (f_{i+1} - 2f_i + f_{i-1}) \\
  f_i''' &= a_3 \frac{h^2}{h} (f_{i+1} - f_{i-1}) + b_3 \frac{h}{h} (f_{i+1}''' - f_{i-1}'') + c_3 (f_{i+1}''' + f_{i-1}''') = \frac{d_3}{h^3} (f_{i+1} - f_{i-1})
\end{align*}
\]

- **Spectral like CD** (Lele, 1992)
- **Spectral like CCD** (Nichi & Ishii, 2003)
  -> achieves high resolution
Spectral like Combined Compact Difference scheme (Sp-CCD)  
(Nihei & Ishii, 2003)

\[ \mathbf{A} \begin{pmatrix} f'_{i-1} \\ f''_{i-1} \\ f'''_{i-1} \end{pmatrix} + \mathbf{E} \begin{pmatrix} f'_{i} \\ f''_{i} \\ f'''_{i} \end{pmatrix} + \mathbf{B} \begin{pmatrix} f'_{i+1} \\ f''_{i+1} \\ f'''_{i+1} \end{pmatrix} = \mathbf{g}_i \left( f'_{i-1}, f'_{i}, f'_{i+1} \right) \]

\[ \text{SH spherical harmonics spectral model} \]
(ispack-0.5; Ishioka, 2000)

Tri-diagonal block matrix form

\[ \begin{bmatrix} \mathbf{E} & \mathbf{B} & \mathbf{A} \\ \mathbf{A} & \mathbf{E} & \mathbf{B} \\ & & \ddots & \ddots & \ddots \\ \mathbf{B} & & \mathbf{A} & \mathbf{E} & \mathbf{B} \\ & & & \mathbf{A} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_{N-1} \\ \mathbf{X}_N \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{N-1} \\ \mathbf{g}_N \end{bmatrix} \]

\[ \mathbf{X}_i = \begin{pmatrix} f'_{i} & f''_{i} & f'''_{i} \end{pmatrix} \]

High accuracy  
High resolution  
Parallel efficiency  

1PE of Fujitsu  
VPP-5000
2. Experimental setup

**Basic equation (SH equation on a rotating sphere)**

\[
\begin{align*}
\frac{\partial u}{\partial t} + v \cdot \nabla u - (f + \frac{u}{a} \tan \theta)v + g \frac{\partial h}{a \cos \theta \partial \lambda} + \alpha(u - \bar{u}) &= 0 \\
\frac{\partial v}{\partial t} + v \cdot \nabla v - (f + \frac{u}{a} \tan \theta)u + g \frac{\partial h}{a \partial \theta} + \alpha v &= 0 \\
\frac{\partial h}{\partial t} + v \cdot \nabla h + \frac{h}{a \cos \theta} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \theta}{\partial \theta} \right) &= 0
\end{align*}
\]

\( \lambda \): latitude  \( \theta \): longitude  
\( u, v \): velocity  \( h = \eta + H \): depth of fluid  
\( f \): Coriolis frequency  
\( g \): gravitational acceleration  
\( a \): earth radius  \( \alpha \): forcing parameter

**Basic state (unstable zonal jet)**

\[
\bar{\eta} = -\frac{fBU_0}{g} \tan \theta \left( e^{\frac{2(\theta - \theta_j)}{B}} \right)
\]

\[
\bar{u} = -\frac{af}{\tan \theta} \pm \sqrt{\left( \frac{af}{\tan \theta} \right)^2 + \frac{8fBU_0}{\tan \theta} \left( e^{\frac{2(\theta - \theta_j)}{B}} \right) \left( \frac{1 + e^{\frac{2(\theta - \theta_j)}{B}}}{\tan \theta} \right) + \frac{4fBU_0 \cos \theta}{\tan \theta} \tan \theta \left( e^{\frac{2(\theta - \theta_j)}{B}} \right) }
\]

\( B \): width of jet  
\( U_0 \): intensity of jet  
\( \theta_j \): latitude of jet
Condition of numerical simulation (using CCD)

**Resolution**: $(\lambda, \theta) = 512 \times 256$ grids

**Boundary condition**: no grid at pole, periodic boundary

**Numerical filter**: low pass filter

**Time integration**: 4th-order Runge-Kutta (full explicitly)

**Experimental parameter**:

- $f_{\text{mid}} = 2\Omega \sin(\pi/4)$, $U_{\text{max}} = 100$
- $g = 9.806$, $a = 6.317 \times 10^6$, $B = 2 \times 10^5$

$$Ro = \frac{U_{\text{max}}}{f_{\text{mid}} B} \approx 1.5 - 30 \quad \text{Sweep } \Omega$$

$$Fr = \frac{U_{\text{max}}}{\sqrt{gH_0}} \approx 0.7$$

$$\theta_j = 11.25 - 78.75, \quad \text{Sweep latitude of jet}$$
3. Results

- Time evolution of flow field

\[ dt = 4\text{s}, \ 300000\text{steps}, \ about \ 14\text{days} \]

- \( Ro=10, \ \theta_j=45 \)

30 hours calculation in 16 cpu of Fujitsu prime power HPC-2500
Latitudinal dependence of flow fields at high $\text{Ro}(=10)$

- $\theta_j = 67.5$
- $\theta_j = 45$
- $\theta_j = 22.5$

Wave number of vortices decreases at higher jet latitude
Latitudinal dependence of GW flux at high Ro(=10)

Zonal mean gravity wave flux

Ro=10, Fr=0.7

Total amount of GW (source) decreases moderately due to shortening of longitudinal length at higher latitude jet.
Latitudinal dependence of flow fields at low Ro(=1.5)

- $\theta_j = 67.5$
- $\theta_j = 45$
- $\theta_j = 22.5$

Significant change of GW propagation and radiation
Latitudinal dependence of GW flux at low Ro (=1.5)

Zonal mean gravity wave flux

Ro = 1.5, Fr = 0.7

Latitudinal changes of $f$ cause
1. No GW radiation from high latitude jet
2. No GW propagation from low latitude jet to high latitude
GW source with fixed $f$ at the jet (Lighthill theory; Ford, 1994)

\[
\left(\frac{\partial^2}{\partial t^2} + f^2 - gH_0 \Delta\right) \frac{\partial h}{\partial t} = \frac{1}{\alpha^2 \cos \theta} \left[ \frac{\partial}{\partial \lambda} \left( \frac{\partial (fu^2)}{\partial \lambda} \right) + \frac{\partial^2}{\partial \lambda \theta} \left( \frac{\partial (2hu^2)}{\partial t} \right) + \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial \theta} \left( \frac{\partial (hu^2)}{\partial t} \right) \right) \right]
\]

\[
+ \frac{1}{\alpha^2 \cos \theta} \left[ \frac{\partial}{\partial \lambda} \left( -fh\omega \right) + \frac{\partial^2}{\partial \lambda \theta} \left( fh\omega^2 - fhu^2 \right) + \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial \theta} \left( fh\omega \right) \right) \right]
\]

\[
+ \frac{1}{\alpha^2 \cos \theta} \left[ \frac{\partial}{\partial \lambda} \left( \frac{g}{2} \frac{\partial}{\partial t}(h - H_0)^2 \right) + \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial \theta} \left( \frac{g}{2} \frac{\partial}{\partial t}(h - H_0)^2 \right) \right) \right] + F \text{(forcing term & dumping)}
\]

\[Ro = 10, Fr = 0.7\]

Source related to vortex is dominant but that related to Coriolis is not negligible
Frequency spectra of GW source

GW source is not so much affected by the changes of parameter

Inertial cut-off of GW radiation and propagation
4. Summary

- We investigate spontaneous gravity wave radiation from unsteady jet flows in shallow water system on a rotating sphere, using combined compact difference scheme.

- Spontaneous gravity waves radiation from unsteady jet flows on the northern hemisphere are observed on the southern hemisphere.

- Gravity wave flux depends on latitude of the jets, since the effects of the earth rotation and size are different.

- Gravity wave source and its analysis on the basis of $f$-plane approximation is useful to understand spontaneous gravity wave radiation from rotational flows.

This work was supported by a Grant-in-Aid for the 21st Century COE “Frontier of Computational Science” and Young Scientists (B) (19740290) from the Ministry of Education Culture, Sports, Science and Technology in Japan.
Future work

- 2 layer shallow water model (collaborated with K. Ishii)
  (Spontaneous internal GW radiation from baroclinic unstable jet)

Parameterizations of spontaneous GW radiation
Analytical estimation (collaborated with H. Kobayashi & Y. Shimomura)

Matched asymptotic expansion for Ro>1, Fr<<1 (Ford et al., 2000)

\[
\phi_{\text{time-dependent}}^2 (x, t) = -\frac{f}{8\pi} \left( \frac{x_i x_j}{r^2} - \frac{\delta_{ij}}{2} \right) \frac{d}{dt} \iint_{R^2} x'_i x'_j \hat{q}(x', t) d^2 x'
\]

\[
- \frac{1}{8\pi} \left( \frac{x_i x_j}{r^2} - \frac{\delta_{ij}}{2} \right) \frac{d}{dt} \iint_{R^2} x'_i x'_j \nabla_{x'} \cdot (\hat{q}(x', t) \nabla_{x'} \psi_0 (x', t)) d^2 x'
\]

\[
\psi_{\text{time-dependent}}^2 (x, t) \sim + \frac{f^2}{8\pi} \left( \frac{x_i x_j}{r^2} - \frac{\delta_{ij}}{2} \right) \iint_{R^2} x'_i x'_j \hat{q}(x', t) d^2 x'
\]

\[
+ \frac{f}{8\pi} \left( \frac{x_i x_j}{r^2} - \frac{\delta_{ij}}{2} \right) \frac{d}{dt} \iint_{R^2} x'_i x'_j \nabla_{x'} \cdot (\hat{q}(x', t) \nabla_{x'} \psi_0 (x', t)) d^2 x'
\]

Estimation for far field: vortex pair, Kirchhoff ellipse…
Validation of analytical estimation using numerical simulation
Parameter sweep experiment to focus on the effect of the Earth rotation
References

Balanced flow

Balance regimes for the stability of a jet in an $f$-plane shallow water system,
Norihiro Sugimoto, Keiichi Ishioka, and Shigeo Yoden,

Spontaneous GW radiation

- $Fr$ dependencies (f-plane SH)
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- Parameter sweep experiments (f-plane SH)
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