Internal Waves and Vortices in Rotating, Stratified Protoplanetary Disks:
Their Roles in Star and Planetesimal Formation

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Jiang & Marcus PRL 2009
Asay-Davis, Thesis; Deck, thesis
Observations of Protoplanetary Disks

Mass $0.01 - 0.1 \, M_{\text{sun}}$
Diameter $\approx 100 - 1000 \, \text{AU}$
Age $\leq 10 \, \text{million years}$
Fluid Dynamics (along with radiation) Determines Transport Properties

Angular momentum
Dust Grains
Migration of Planetesimals and Planets

No observations of turbulence or fluid structures (yet)
Eddy Viscosity $\nu_{\text{eddy}}$

* Replace the nonlinear, and difficult-to-calculate advective term – $(\mathbf{V} \cdot \nabla)\mathbf{V}$ with a ficticious, linear, easy-to-calculate diffusion $\nabla \cdot \nabla (\nu_{\text{eddy}} \mathbf{V})$

* Set $\nu_{\text{eddy}} = \alpha c_s H_0$  
  (Shakura & Sunyaev 1973)

$\alpha < 1$ because turbulent eddies are probably subsonic and not larger than a scale height $H_0$ in extent.

* Parameterize angular mom. transfer and/or rate of mass accretion in terms of $\alpha$.

Origin of turbulence? Shear instabilities, convection, MHD instabilities … etc.
Does $v_{\text{eddy}}$ work for Angular Momentum Transport?

- Successful with heat transfer in non-rotating and non-shearing flows such as thermal convection, e.g., Prandtl mixing-length theory.
- Runs into problems when used for transporting vectors quantities and when there is competition among “advectively conserved” quantities. e.g., Spherical Couette Flow
- Do not violate Fick’s Law.
  Angular momentum angular must be transported radially outward from a forming protostar; yet the disk’s ang. mom. increases with radius.
Formation of Planetesimals

• 2 Competing Theories
  – Binary Agglomeration: Sticking vs. Disruption?
  – Gravitational Instability: Settling vs. Turbulence?
  
  Toomre criterion: \( v_d < \frac{\pi G \Sigma_d}{\Omega_K} \approx 10 \text{ cm/s} \)
Vortices in Protoplanetary Disks?

Recipe for vortices:
- Rapid rotation
- Intense shear
- Strong stratification
Vortices in Protoplanetary Disks?

Anticyclonic shear leads to anticyclonic vortices
Base Flow in Protoplanetary Disk

Disk is Cold:

- Pressure small enough that radial pressure gradient
- Gas is un-ionized and therefore not coupled to magnetic field
- Sound speed $c_s \ll$ Keplerian velocity $V_K$
Base Flow in Protoplanetary Disk

Near balance between gravity and centrifugal force:

\[
\bar{u}_\phi = \sqrt{\frac{GM}{r}} \left( 1 - \eta(r, z) \right)
\]

\eta \approx 10^{-3}

\[
\sigma_k \equiv r \frac{\partial}{\partial r} \left( \frac{v_k}{r} \right) = -\frac{3}{2} \frac{v_k}{r} = -\frac{3}{2} \Omega_k
\]

Anticyclonic Shear

No hydrostatic balance in the radial direction!
Base Flow in Protoplanetary Disk

Vertical hydrostatic balance:  
\[- \frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} = \frac{GM}{r^2} \frac{z}{r} \]

Cool, thin disk:

\[c_s \approx \Omega_k H_0\]

\[\delta \equiv \frac{H_0}{r_0} \approx \frac{c_s}{v_k} \ll 1\]
Vortices in Protoplanetary Disks

• Shear will tear a vortex apart unless:
  – Vortex rotates in same sense as shear. In PPD, vortices must be ANTICYCLONES.
  – Strength of the vortex is at least of the same order as the strength of the shear.

• $\omega_V \sim \sigma_k \sim \Omega_k$

• $Ro \equiv \frac{\omega_V}{2\Omega_k} \sim 1$
Vortices in Protoplanetary Disks

• Velocity across vortex must be subsonic; otherwise sound waves & shocks would rapidly dissipate kinetic energy of vortex:
  – $V \sim \sigma_k L < c_s$
  – $V \sim \Omega_k L < c_s$
  – $\varepsilon \equiv V/c_s \sim (\Omega_k/c_s) L < 1$
  – But from hydrostatic balance: $c_s \sim \Omega_k H$
  – $\varepsilon \equiv V/c_s \sim L/H < 1$
### PPD (L ≈ H) vs. Jovian Vortex (L >> H)

<table>
<thead>
<tr>
<th>Timescales</th>
<th>Jovian Vortex</th>
<th>PPD</th>
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</thead>
<tbody>
<tr>
<td>$\tau_{\text{turn}} \equiv 2\pi/\omega_V$</td>
<td>~ 50 h.</td>
<td>~ 1 y.</td>
</tr>
<tr>
<td>$\tau_{\text{sh}} \equiv 2\pi/\sigma$</td>
<td>~ 20 h.</td>
<td>~ 1 y.</td>
</tr>
<tr>
<td>$\tau_{\text{rot}} \equiv 2\pi/\Omega$</td>
<td>~ 10 h.</td>
<td>~ 1 y.</td>
</tr>
<tr>
<td>$\tau_{\text{bv}} \equiv 2\pi/N$</td>
<td>~ 10 m.</td>
<td>~ 1 y.</td>
</tr>
<tr>
<td>Rossby Ro $\equiv \tau_{\text{rot}}/2\tau_{\text{turn}}$</td>
<td>~ 0.2</td>
<td>~ 1</td>
</tr>
<tr>
<td>Fr $\equiv (\tau_{\text{bv}}/\tau_{\text{turn}}) (L/H)$</td>
<td>~ 0.25</td>
<td>~ 1</td>
</tr>
<tr>
<td>Richardson $\equiv 1/\text{Fr}^2$</td>
<td>~ 16</td>
<td>~ 1</td>
</tr>
</tbody>
</table>
Key to the Asymptotics

• Gravity sets rotation, shear & stratification:
  – \( \Omega_0 \equiv (GM/r_0^3)^{1/2} \)
  – Rotation: \( = 2\Omega_0 \)
  – Shear: \( \sigma = (3/2)\Omega_0 \)
  – Stratification: \( \omega_{bv} \approx 2\Omega_0 \)
  – Vorticity: \( \omega_z \approx \sigma \approx (3/2) \Omega_0 \)

• Very different from Jupiter
Equations of Motion

• Momentum: With Coriolis and Buoyancy
\[
\frac{\partial v}{\partial t} = -v \cdot \nabla v - \nabla h - 2\Omega_0 \hat{z} \times (v - \bar{v}) + \frac{T}{T_0} \Omega_0^2 \hat{z} \hat{z}
\]

• Divergence: Anelastic Approximation
\[
\nabla \cdot \left( \bar{\rho}v \right) = 0
\]

• Temperature: With Pressure-Volume Work
\[
\frac{\partial T}{\partial t} = -v \cdot \nabla T - (\gamma - 1)T_0(\nabla \cdot v) - \frac{T - T_0}{\tau_{cool}}
\]
Two-Dimensional Approx. is not correct and misleading

Too easy to make vortices due to limited freedom, conservation of $\omega$ and inverse cascade of energy
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Computational Method

• Cartesian Domain: \((r,\phi,z) \rightarrow (x,y,z)\)
  – Valid when \(\delta \equiv H_0/r_0 \ll 1\) and \(\Delta r/r_0 \ll 1\)

• 3D Spectral Method
  – Pseudo-Lagrangian or sheared coordinates in \(x,y\)
  – Horizontal basis functions: Fourier-Fourier basis
  – Vertical basis functions: Chebyshev polynomials for truncated domain or Rational Chebyshev functions for infinite domain

• Parallelizes and scales
Tall Columnar Vortex
Tall Columnar Vortex
Vortex in the Midplane of PPD

\[ r_0 \Delta \phi = 4H_0 \]

Blue = Anticyclonic vorticity, Red = Cyclonic vorticity

\[ \Delta r = 2H_0 \]

\[ Ro = 0.3125 \]
Vortex in the Midplane of PPD

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\( \phi-z \) plane at \( r=r_0 \)

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$\Delta z = 8H_0$

$r_0\Delta\phi = 4H_0$
Maximum absolute value of antisymmetric component of $\omega_z$

$\tau_{\text{instability}} \approx 3.05 \tau_{\text{orb}}$
$r-z$ plane at $\phi=0$

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$\Delta z = 8H_0$

$\Delta r = 2H_0$
$r-z$ plane at $\phi=0$

$Ro = 0.3125$

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$\Delta z = 8H_0$

$\Delta r = 2H_0$
Refracting Internal Wave

Beam refracts so that it is vertical when it hits the region where $N(z) = \omega$

Wave velocity increases: kinetic energy flux is nearly constant
$z = 1.5 H_0$

$z = 1.0 H_0$

$z = 0 H_0$
Spontaneous Formation of Off-Midplane Vortices

z-component of vorticity

$\Delta x = 2 \, H_0$

$\Delta y = 8 \, H_0$

1 second = 42 years

blue = anticyclonic
red = cyclonic
Spontaneous Formation of Off-Midplane Vortices

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Refracting Internal Wave

Vertical vorticity is created at \( z \) where \( N = \omega \)
3D Vortex in PPD
Angular Momentum Transport

Transport is proportional to $V_r V_\phi$

Fore-Aft symmetry:

$V_r(\phi) = -V_r(-\phi)$

$V_\phi(\phi) = +V_\phi(-\phi)$
Grain Trapping in Vortices

- Grains feel Coriolis, centrifugal, gravity (from all sources) and gas drag forces.
- Stokes Drag Stopping Time (or use Epstein for $r_{\text{grain}} < \text{mean free path} \approx 1\text{cm at 1AU}$):

\[ f_{\text{drag}} = \frac{(v_{\text{gas}} - v_{\text{grain}})}{t_S} \]

\[ t_S \equiv \frac{1}{9} \frac{\rho_{\text{grain}} r_{\text{grain}}^2}{\rho_{\text{gas}} \nu m} \]

- At 1 AU:

\[ t_S = 100s \left( \frac{r_{\text{grain}}}{1\text{cm}} \right)^2 \]
Why don’t grains centrifuge out of vortices?
Grain Trapping in 2D Vortices

The attractors here are all limit cycles not points.
Trapping Dust in 3D Vortices

Examples:
- Sand storm
- Sand suspended by waves
- Hail

Characterized by
- Primary horizontal flow
- Secondary vertical flow
- Updrafts suspend particles against gravity
- Characteristic times for grains to fall to disk mid-plane
3D Vortex

- High Pressure Center due to Geostrophic Balance between Coriolis and radial pressure.

- Heavy upper lid and buoyant lower bottom are created by the vertical flow and lead to vertical balance (in accord with thermal wind equation).
Equilibrium in Horizontal

- Horizontal momentum equation:

- For $\text{Ro} \ll 1$, Geostrophic balance between gradient of pressure and the Coriolis force.

- Anticyclones have high pressure centers.

- For $\text{Ro} \gg 1$, low pressure centers.
Role of $v_z$

- In sub-adiabatic flow: rising cools the fluid while sinking warms it.
- This in turn creates cold, heavy top lids and warm, buoyant bottom lids.
- This balances the vertical pressure force (and has horizontal temperature gradients in accord with the thermal wind equation).
- Numerical calculations show that after lids are created, $v_z \neq 0$.
- Magnitude of $v_z$ is set by dissipation rate
Are Particles Passive?

Assumption breaks down when:

1. $\rho_{\text{dust}} \sim 1/10 \rho_{\text{gas}}$ – back-reaction on gas
2. $s_{\text{dust}} \sim 20 r_{\text{dust}}$ -- particles feel each other’s wakes
3. Particles collide

• Assumption (1) breaks down first and if a Toomre-like criterion is valid for gravitational collapse, then the collapse occurs before assumptions (2) or (3) fail.

• Toomre criterion suggests $\rho_{\text{dust}} \sim 10^{-7}$ g/cm$^3$, but $\rho_{\text{gas}} \sim 10^{-9}$ g/cm$^3$
Dust Dynamics
Attracting Regions
Dust Dynamics
Attracting Regions
Hollow Vortices
Hollow Vortices
Conclusions

• (Some) vortices are unstable in the midplane (where the stratification vanishes)
• (Some) vortices thrive off the midplane (where there is stratification)
• Vortices from spontaneously from internal gravity waves and draw their energy from the Keplerian shear.
• Vortices transport angular momentum radially outward, leading to accretion
• Vortices capture and concentrate dust grains
Future / Current Work

• Formation Mechanisms
  – Spin-up of temperature & density “lumps”
  – Inverse cascade (small vortices merging into larger vortices)

• Dust dynamics
  – Grain collision rates & velocities
  – Gravitational settling & instability
  – 2-Fluid Model

• Mass and Angular Momentum Transport