Equilibria of corotating nonuniform vortices

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Inviscid equilibria for a pair of two-dimensional vortices of different size and circulation are numerically computed. Previous analyses studied the case of vorticity patches. In the present investigation, we introduce vortex profiles ranging from patches to Lamb-type profiles, and analyze how classical results are altered when the vorticity distribution is not piecewise constant. Furthermore we study how nonuniformity of the vorticity distribution affects the instability threshold which has been previously interpreted as a sign of the merging process. It is shown as well how this instability point is modified when one of the vortices is replaced by a point vortex of identical circulation. For this purpose, a numerical solution procedure has been developed capable of computing the perturbed streamlines, using Green’s function integrals. © 1999 American Institute of Physics. [S1070-6631(99)0211-1]

I. INTRODUCTION

Many numerical,1,2 analytical3 or experimental works4–7 have been devoted to decaying two-dimensional turbulence. Understanding such a process has indeed important consequences for geophysics, meteorology, and astrophysics. Various aspects appear in a recurrent way; from an initially unorganized two-dimensional flow, vorticity rapidly concentrates in localized regions called coherent structures and also in vorticity filaments.8 Jimenez et al.2 have shown by comparison between numerical computations and an asymptotic analysis, that such vortices are fairly well described by perturbed Lamb vortices with elliptical shape due to local strain field. When they are far apart, the evolution of two-dimensional vortices can be hence easily modeled9,10 For instance Melander et al.11 used a moment model. However, Pierrehumbert16 computed the case of two opposite-signed symmetrical vortices, Saffman and Szeto17 worked out the equilibria of two like-signed vortices, and finally Dritschel13,18 considered two unequal-area vortices. These authors meant by vortex a finite region of constant vorticity bordered in an irrotational infinite flow. This approach deeply simplifies the Euler’s equations since time evolution is reduced to contour dynamics, a typical one-dimensional problem. Vortices with piecewise constant vorticity have also been considered where the vortex consists of nested uniform vortices of elliptical shape.9

However, vortices of two-dimensional turbulence are not vorticity patches. Our purpose is thus to verify how previous results obtained in the context of uniform vorticity are modified when Lamb-type vortices are considered. We compute the equilibria between two vortices with different circulation and size as a function of the distance between their centroids. This step is performed by quadrature of Green’s function integrals together with a branch continuation method,19 the unknown of the nonlinear system being the perturbed streamlines. Note that numerical solution methods based on quadrature to determine shapes of equilibrium states have previously been considered, for two-dimensional uniform vortices16 or vortex rings.20 To initiate the continuation procedure we use the following Euler solution: two stationary sequence of a purely inviscid mechanism evolving much faster than viscous diffusion. When viscosity is included, a quasiequilibrium should be considered since slow viscous diffusion will prevent any exact equilibrium to exist (see Ref. 15 for viscous vortices solutions based on a stochastic approach).

In this framework, the computation of a nonlinear equilibrium between two vortices and its stability becomes an essential ingredient. This problem has been repeatedly studied. Dritschel13,18 considered two unequal-area vortices16 or vortex rings.20 To initiate the continuation procedure we use the following Euler solution: two stationary
axisymmetric vortices of given circulation and profile, located far apart. In the following a Gaussian profile typical of the usual Lamb vortex is used. Moreover, we assume that the vorticity field does not spread to infinity but it is hence bounded by a contour. In the inviscid approximation it is indeed possible to cut the exponential tail of vortices. Our continuation parameter is the gap between vorticity centroids. Gradually reducing the distance from infinity, we compute a state formed by the two vortices stationary when viewed in a rotating reference frame. Each vortex is nonaxisymmetric because of the finite-distance vortex interactions and the numerical continuation procedure determines the angular rotation and structures of the vortices for each distance.

As shown in Dritschel,\textsuperscript{18} vortex pairs formed by two patches lose stability for a critical distance which is associated with an exchange of stability. We assume that this characteristic persists for general profiles, which is not completely justified. Our numerical procedure necessitating the computation of the Jacobian matrix, points of exchange of stability. i.e., zero eigenvalues, are detected in the parameter space, as a by-product. In fine a general stability analysis, out of the scope of the present work, should validate this hypothesis.

In Sec. II we present the equations governing the vortex pair dynamics. The numerical procedure used to solve the model is then described in Sec. III. Section IV is devoted to the results of the analysis. Some conclusions are drawn in Sec. V.

II. THE GOVERNING EQUATIONS

We consider a pair of nonaxisymmetric vortices which rotate with an overall rotation rate around a center $O$ located along an axis of symmetry $x$. The geometry of the equilibrium is sketched in Fig. 1(a). The center of the vortices are points of vorticity maximum, they are located, respectively, at $O_1$ and $O_2$ on the $x$-axis. We introduce $L$ the distance between the vortex centers, and $a$ the distance between the center of vortex 1 and the global origin $O$. Let $\omega_1$ (respectively, $\omega_2$) denote the vorticity distribution inside the vorticity region 1 (respectively, region 2). The stream function generated respectively by $\omega_1$ and $\omega_2$ satisfies $\Delta \psi_i = -\omega_i$, $i = 1, 2$, or in terms of Green’s function integral

$$\psi_i(r) = \frac{1}{4\pi} \int \int_{A_i} \log(|r - r'|^2) \omega_i(r')dS,' $$

where $r, r'$ in the above formula are taken from the center of vortex $i$, $i = 1, 2$. In the reference frame $(x, y)$ rotating with $\Omega$, the total stream functions read

$$\psi_{1,2}(r_i) = \psi_1(r_1) + \psi_2(L + r_1) + \frac{1}{2} \Omega |a + r_1|^2$$

and

$$\psi_{1,2}(r_2) = \psi_1(r_2) + \psi_2(-L + r_2) + \frac{1}{2} \Omega |L + a + r_2|^2.$$

In (2), (3) $a = \overrightarrow{O_1} = (a, 0)$, $L = \overrightarrow{O_2O_1} = (L, 0)$ and $r_i$ are the coordinate vectors with origin at $O_i$, $i = 1, 2$ [see Fig. 1(a)]. When vortices are far apart, their mutual interaction is quite weak and the undisturbed vortices are naturally supposed to be axisymmetric (although, as recently pointed out by Dritschel,\textsuperscript{21} isolated smooth vortices which sufficiently steep gradients are not necessarily axisymmetric). The problem is hence obviously degenerated. Any axisymmetric vorticity distribution is indeed possible. Each will generate a possible family of solutions when vortices are brought at a finite distance. In order to remove this degeneracy, we pick up a given profile for infinitely remote vortices. Owing to the numerical findings of Jimenez et al.\textsuperscript{2} we are interested in axisymmetric Lamb-type vortices

$$\omega_i^* = \omega_{\max, i} \exp(-\alpha \rho_i^2/r_{\max, i}^2),$$

$$0 \leq \rho_i^* \leq r_{\max, i}^*, \quad i = 1, 2.$$  (4)

Our vortices being of finite extent the parameter $\alpha$ measures the ratio between the vorticity maximum located at the center and the minimum vorticity located at the boundary for each vortex, that is

$$\omega_{\min, i}^* = \omega_{\max, i} e^{-\alpha}, \quad i = 1, 2.$$  (5)

Using Lamb-type vortices to initiate the computation of equilibria, the vorticity distribution is smooth inside the vortex whereas at the edge there is a jump condition. Regions with Gaussian-type vorticity distributions have been found in decaying two-dimensional turbulence\textsuperscript{6} and Lamb-type vortices are hence relevant for the purpose of the present analysis. An alternative would be to consider axisymmetric vortices of compact support\textsuperscript{12} indeed, our solution technique works whenever the radii of the undisturbed streamlines inside each vortex are function of the vorticity. Even though it would be preferable to consider a continuous vorticity decrease to zero rather than a jump condition, one may also argue that the lowest vorticity levels would not survive, as a consequence of a stripping process.\textsuperscript{10}
The system is made dimensionless in such a way that vortex 1 has a circulation \( \Gamma_1 = \pi \) and an area \( A_1 = \pi \). This implies that \( r_{\max,1}^* \) is the reference length and \( \omega_{\max,1}^* \beta \) with \( \beta = (1 - e^{-\alpha})/\alpha \) the reference vorticity. The dimensionless vorticity distribution inside the undisturbed vortex 1 is

\[
\omega_1(r) = \frac{1}{\beta} \exp(-\alpha r^2), \quad 0 \leq r \leq 1,
\]

\( A_1 = \pi, \quad \Gamma_1 = \pi. \)  

When written in dimensionless form the vorticity distribution of the undisturbed vortex 2 now reads

\[
\omega_2(r) = \frac{\alpha \omega}{\beta} \exp(-\alpha r^2), \quad 0 \leq r \leq 1/\alpha_r, \quad A_2 = \frac{\pi}{\alpha_r}, \quad \Gamma_2 = \frac{\alpha \omega \pi}{\alpha_r},
\]

where

\[
\alpha \omega = \omega_{\max,2}^* \omega_{\max,1}^* \quad \text{and} \quad \alpha_r = r_{\max,1}^*/r_{\max,2}^*.
\]

A typical vorticity distribution along the \( x \)-axis for undisturbed vortices is sketched in Fig. 1(b), for \( A_2/A_1 = 0.5 \), that is \( \alpha_r = \sqrt{2} \), with \( \alpha_\omega = 1 \) and \( \alpha = 2.25 \). For this example the vorticity at the boundary of each vortex is approximately one-tenth of the vorticity at the center.

For the computation of the stream functions (2), (3), integrals of the form (1) have to be evaluated inside the areas \( A_1 \) and \( A_2 \) of the vortices. For an equilibrium solution of the Euler’s equations the isovorticity contours are streamlines. When the vortices are far apart, the radius of a given circular streamline in each vortex is function of vorticity. By inversion of (6) and (7) one gets

\[
r_{\max,1}(\omega_1) = \frac{1}{\alpha} \log(\beta \omega_1), \quad 1/\beta \gg \omega_1 \approx e^{-\alpha}/\beta,
\]

\[
r_{\max,2}(\omega_2) = \frac{1}{\alpha \omega_r} \log(\beta \omega_2/\alpha_\omega), \quad \alpha_\omega/\beta \gg \omega_2 \approx e^{-\alpha} \alpha_\omega/\beta.
\]

When the vortices are at a finite distance from each other, the streamlines are deformed. Introducing the functions \( f_i(\omega_i, \theta), \ i = 1,2 \), the same constant vorticity \( \omega_1 \) (respectively, \( \omega_2 \)) now lies on the perturbed streamlines

\[
r_i(\omega_i, \theta) = \sqrt{r_{\max,1}^2(\omega_i) + f_i(\omega_i, \theta), \quad 0 \leq \theta \leq 2 \pi, \quad i = 1,2.}
\]

The vectors \( \mathbf{r}_i \) are expressed from the vortex centers \( O_i \), where vorticity is maximum. Hence we have by construction

\[
r_i(\omega_i, 1) = 0 \quad (\omega_i \text{ being the maximum vorticity}) \quad \text{and according to} \quad (10) \quad \text{and} \quad (9),
\]

\[
f_i(\omega_i, 1) = 0, \quad 0 \leq \theta \leq 2 \pi, \quad i = 1,2.
\]

Using the relation (10) inside each vortex, Green’s function integrals (1) are expressed as vorticity integrals

\[
\psi_i(\mathbf{r}) = -\frac{1}{4\pi} \int_{\omega_{i,0}}^{\omega_i} \int_0^{2\pi} \log(|\mathbf{r} - \mathbf{r}_i(\omega_i, \theta')|^2) \frac{1}{2} \omega_i' \times \left( -\frac{\partial^2}{\partial \omega_i} (\omega_i', \theta') \right) d\omega_i' d\theta',
\]

the vorticity ranging from its minimum value \( \omega_{i,0} \) to its maximum value \( \omega_i \) for each vortex region \( i = 1,2 \). Note that \( \partial r_{\max,i}^2/\partial \omega_i < 0 \); when the vortices get closer (at least for the equilibrium solutions we are looking for) we suppose that two distinct streamlines never cross each other and hence \( \partial r_{\max,i}^2/\partial \omega_i < 0 \) also for the disturbed streamlines.

The vectors \( \mathbf{r}_i = (r_i \cos(\theta), r_i \sin(\theta)) \) can be written in terms of the vorticity \( \omega_i \) and the angle \( \theta \) using (10). Consequently, the condition that the total stream function (2) (respectively (3)) is a constant on each vorticity level inside vortex 1 (respectively vortex 2) can now be written as

\[
\psi_i(\omega_i, \theta) = C_i(\omega_i), \quad 0 \leq \theta \leq 2 \pi, \quad i = 1,2,
\]

the integrals being evaluated using (12). Equation (13) can be viewed as nonlinear constraints for the unknowns \( f_i(\omega_i, \theta), \ i = 1,2 \).

The underlying assumption for our computations is, that infinitely remote vortices, when drifting slowly toward each other under the action of a weak external field, undergo a sequence of quasiequilibrium states. The evolving vortex shapes are assumed to depend merely on the mutual interaction between both vortices. Hence each vortex region should preserve its area \( A_i \) as well as an infinite number of dynamical invariants of inviscid fluid motion

\[
\int \int_{A_i} \nu \partial S_i, \quad p = 1,2, ...
\]

when the distance between the two vortices is modified. These conditions are automatically satisfied if

\[
\int_0^{2\pi} f_i(\omega_i, \theta) d\theta = 0, \quad \omega_{i,0} \approx \omega_i \approx \omega_{i,1}, \quad i = 1,2.
\]

The family of solutions is supposed to be stationary in the frame of reference rotating with \( \Omega \). For a general equilibrium configuration the angular rotation rate \( \Omega \) as well as the distance \( a \) between the center of rotation and the center of vortex 1 should be determined as function of the distance between the vortex centers \( L \), which is our continuation parameter of the nonlinear system. Two extra-conditions are obtained by letting the centers of both vortices to be stagnation points in the rotating frame of reference. The vortices being symmetric with respect to the \( x \)-axis, it follows that the stream functions (2), (3) must satisfy

\[
\frac{\partial \psi_i}{\partial \theta}(r_i, \theta = 0) = 0.
\]

Consequently, the centers of the vortices to be stagnation points leads to a single condition for each vortex

\[
\lim_{r_i \to 0} \frac{\partial \psi_i}{\partial r_i}(r_i, \theta = 0) = 0, \quad i = 1,2.
\]

Using the expression of the total stream functions (2), (3), and transforming integrals

\[
\frac{\partial \psi_i}{\partial r_j}(\mathbf{r}) = -\frac{1}{2\pi} \int \int_{A_i} \frac{(\mathbf{r} - \mathbf{r}_j') \partial \mathbf{r}_i / \partial \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j'|^2} \omega_j(\mathbf{r}_j') dS_i'
\]

by (10), these two conditions read
\[ \frac{\partial \psi_{i,j}}{\partial r_i}(\omega_{i,1}, \theta = 0) = 0, \quad i = 1, 2. \] (16)

Summarizing, the family of equilibrium states parameterized by \( L \) is characterized by the nonlinear system (11),(13),(14), involving integrals of type (12), the overall rotation rate \( \Omega \) and the distance \( a \) being fixed adding extra-conditions (16).

### III. NUMERICAL SOLUTION PROCEDURE

By decreasing the distance \( L \) between the centers of the initially axisymmetric vortices, the increasing deformation of the streamlines is characterized by the unknown functions \( f_i(\omega, \theta) \), \( \omega \leq \omega_i \leq \omega_{i,1} \), \( 0 \leq \theta \leq 2\pi \), \( i = 1, 2 \). As mentioned in the previous section, the streamlines inside each vortex are symmetric with respect to the \( x \)-axis. Consequently, only the upper half of each constant vorticity contour will be discretized. A Fourier-collocation method is used for the \( \theta \) variable, whereas the \( \omega_i \) variable is transformed into \(-1 \leq \xi \leq 1\) in order to use a Chebyshev-collocation method. The unknowns of the discretized problem are

\[
\begin{align*}
  f_1(\xi_{1,j}, \theta_{1,k}) & : \quad 0 \leq j \leq n_1, \quad 0 \leq k \leq m_1; \\
  f_2(\xi_{2,j}, \theta_{2,k}) & : \quad 0 \leq j \leq n_2, \quad 0 \leq k \leq m_2;
\end{align*}
\]

with

\[
\xi_{i,j} = \cos(j \pi/n_1); \quad \theta_{i,k} = 2\pi k/2m_i; \quad i = 1, 2.
\] (17)

Note that the number of collocation points is not the same for the two vortices. The derivatives with respect to \( \omega_i \) are performed analytically for \( r_i \), whereas the derivatives for the unknown functions (17) are computed using the collocation matrix method.\(^{22}\) For evaluation of the integrals of type (12) Gaussian quadrature is used in the periodic direction \( \theta \) (together with the symmetry condition \( f_i(\xi_{i,j}, \theta_{i,2m_i-k}) = f_i(\xi_{i,j}, \theta_{i,k}) \), \( 0 \leq k \leq m_i \), \( i = 1, 2 \)). A trapezoidal integration scheme is used for the \( \omega_i \) integration. We use this robust though simple method since our numerical experiences led to the conclusion that accuracy can hardly be improved using Gaussian-type quadrature for the Chebyshev-collocation points. Indeed, the integrand has poor regularity, and even a logarithmic singular part. This singular part in the total stream function (13) comes from the contribution \( \psi_i(\omega_i, \theta) \) in the decomposition (2),(3). Using quadrature in \( \xi \) the discretized version of the corresponding integral reads

\[
\psi_i(\xi_{i,j}, \theta) = \psi_i - \frac{\gamma_j}{4\pi} \int_0^{2\pi} \log(1 - \cos(\theta - \theta')) \varphi(\xi_{i,j}, \theta') d\theta',
\] (18)

where \( \varphi(\xi_{i,j}, \theta') \) stands for the remaining regular factor of the integrand of type (12) and \( \gamma_j \) is the coefficient of the quadrature rule. The regular part, that is the sum over \( k \neq j \) in the quadrature formula, is written as \( \psi_i \). Let us add and subtract the integral

\[
I = \frac{\gamma_j}{4\pi} \int_0^{2\pi} \log(1 - \cos(\theta - \theta')) \varphi(\xi_{i,j}, \theta') d\theta' \] (19)

to Eq. (18). The logarithm thus becomes

\[
\log \left( \frac{|r_i(\xi_{i,j}, \theta) - r_i(\xi_{i,j}, \theta')|^2}{1 - \cos(\theta - \theta')} \right),
\]

which has a limit when \( \theta \rightarrow \theta' \). The integral (19) can be evaluated using the change of variable \( u = -\theta + \theta' \) and then by an integration by parts

\[
I = \frac{\gamma_j}{4\pi} \left[ u \log(1 - \cos(u)) \varphi(\xi_{i,j}, u + \theta) \right]_\pi - \frac{\gamma_j}{4\pi} \int_\pi^- \int_0^\pi \frac{u \sin(u)}{1 - \cos(u)} \varphi(\xi_{i,j}, u + \theta) du + u \log(1 - \cos(u)) \frac{\partial \varphi}{\partial u}(\xi_{i,j}, u + \theta) du.
\]

The integrand is now defined for \( u \rightarrow 0 \). Note that the derivatives with respect to \( u \) for the unknown \( \varphi \) are again evaluated by a Fourier-collocation matrix method.

The conditions (16) for the vortex centers to be stagnation points necessitate in particular the evaluation of the integrals

\[
\frac{\partial \psi_i}{\partial r_i}(r_i = 0) = - \frac{1}{4\pi} \int_{\omega_{i,0}}^{\omega_{i,1}} \int_0^{2\pi} \cos(\theta) \omega_i \frac{\partial r_i}{\partial \omega_i}(\omega, \theta) d\omega d\theta, \quad i = 1, 2,
\]

which are nonzero here since the perturbed vortex is nonaxisymmetric. The quantity \( \partial r_i/\partial \omega_i \) is singular at \( \omega_{i,1} \) (see (11)) and we again perform an integration by parts with respect to \( \omega_i \) to recover a regular integrand.

Once discretized conditions (13) become

\[
\psi_i(\xi_{i,j}, \theta_{i,k+1}) - \psi_i(\xi_{i,j}, \theta_{i,k}) = 0, \quad 1 \leq j \leq n_1, \quad 0 \leq k \leq m_i - 1, \quad i = 1, 2.
\] (20)

(Note that \( \xi_{i,0} = 1 \) corresponds to \( \omega_{i,1} \), that is the maximum vorticity level at the center of each vortex.) The discrete version of (11) reads

\[
f_i(1, \theta_{i,k}) = 0, \quad 0 \leq k \leq m_i, \quad i = 1, 2,
\]

and (14) becomes

\[
\int_0^{2\pi} f_i(\xi_{i,j}, \theta) d\theta = 0, \quad 1 \leq j \leq n_1, \quad i = 1, 2.
\] (22)

Conditions (20)–(22) give rise to \((n_1 + 1)(m_1 + 1) + (n_2 + 1)(m_2 + 1)\) equations for the same number of unknowns (17). Finally, conditions (16) provide two more equations to determine \( \Omega \) and \( a \). The nonlinear system (20)–(22) together with (16) is solved using a modified Newton method known as the Broyden algorithm (see Stoer and Bulirsch,\(^{23}\) p. 283). During the continuation procedure, the Jacobian matrix of the nonlinear system has to be updated for each set of iterations. It is not pertinent, in the present case, to seek for an explicit expression for the partial derivatives with respect to the unknowns; these quantities are hence numerically computed by difference quotients. In order to avoid the singularities of ordinary parameter continu-
ation at limit points, Keller’s arclength continuation procedure\(^9\) is used. Points of exchange of stability are detected computing the eigenvalues of the Jacobian matrix. Indeed, the appearance of a zero eigenvalue in the spectrum of the Jacobian matrix is characteristic for exchange of stability. In order to obtain reliable results, the accuracy of the numerically computed Jacobian matrix has been controlled throughout the computations by conveniently choosing the step sizes for the difference quotients.

The family of equilibrium states is parametrized by \(L\). The results are discussed in terms of the distance \(l\) between the centroids. Once an equilibrium state obtained, the centroid is easily computed by evaluating the integral

\[
\frac{1}{\Gamma_1} \int \int_{A_j} r_j \cos(\theta) \omega_j dS_j.
\]

Lamb-type vortices (6), (7) are provided as a quasiexact solution for large \(L\); in this case the vortices are approximately axisymmetric and Eqs. (16) reduce to

\[
-\frac{\Gamma_2}{2\pi} + \Omega a = 0; \quad \frac{\Gamma_1}{2\pi} + \Omega(a - L) = 0.
\]

Note that these conditions are equivalent to those for a pair of point vortices to be stationary in a rotating frame of reference. Starting for a sufficiently large \(L\) value with this asymptotic rotation rate \(\Omega = (\Gamma_1 + \Gamma_2)/2\pi L^2\), distance \(a = L\Gamma_2/(\Gamma_1 + \Gamma_2)\) and with the axisymmetric vortex configuration (6),(7), a first deviation (17) from the axisymmetric case is computed by solving the nonlinear system. The whole family of equilibrium states is then computed step by step by decreasing the continuation parameter \(L\).

IV. RESULTS

Different cases are considered according to the circulation ratio \(\alpha_{\omega}\), area ratio \(\alpha_r\), as well as the quantity \(\alpha\) (see (4)) which characterizes the nonuniformity of the vortex profiles. In Dritschel\(^{18}\) the normalization is such that the total circulation is equal to \(\pi\). For comparison, in all subsequent results the distance \(l\) is multiplied by the scale factor \(\sqrt{\pi/(\Gamma_1 + \Gamma_2)}\), where \(\Gamma_1 + \Gamma_2\) is the total circulation in our computations (cf. Sec. II).

A. Equilibrium states of finite vortices

1. Two identical vortices

Focusing first on two identical vortices, decreasing the numerical continuation parameter \(L\), the isocontours of vorticity (or equivalently the streamlines) are deformed until the vortices almost touch. For vanishing \(\alpha\)-values (cf. (4)) we asymptotically recover the case of patch vortices which has extensively been studied in previous investigations. Figure 2 depicts such a case with \(\alpha = 0.001\). For each vortex the (almost constant) vorticity distribution has been discretized using 6 points and half of each contour is represented using 40 points. (For this almost uniform case only the outer boundary for each vortex is shown.) Even though \(\alpha\) is small contrary to contour dynamics computations we have to take into account various vorticity levels. (The case \(\alpha \to 0\) is a degenerate case for our method.) Keeping the number of vorticity levels constant the use of 60 points for half a contour hardly affects the results. For this uniform case there is evidence of a cusp formation at the boundary point where the vortices almost touch; it has been shown by Overman\(^{24}\) that the corner angle for uniform vortices is equal to \(\pi/2\) in the limit of vanishing gap \(\delta\) between the vortices.

The equilibrium configuration shown is unstable as seen in Fig. 3, where \(\delta\) (the gap between the vortices) is depicted as function of \(l\), the distance between the centroids. The solid line in Fig. 3 corresponds to the almost uniform case (\(\alpha = 0.001\)), the circle corresponding to the appearance of the zero eigenvalue. To be more specific the instability point is located at \(l = 2.26\) and \(\delta = 0.34\), which agrees with the values of Fig. 4 in Dritschel\(^{18}\). The value given in Saffman\(^{14}\) (p. 180) for a highly accurate computation is \(l = 2.243\). The difference is due to the small but finite \(\alpha\) value and the different algorithms used. In the extreme configuration where the vortices almost touch Saffman gives the same \(l\)-value as that for

\[
\begin{align*}
\textbf{FIG. 2.} & \quad \text{Highly unstable equilibrium solution for almost uniform (}\alpha = 0.001\text{) equal vortices.} \\
\textbf{FIG. 3.} & \quad \text{Gap } \delta \text{ as function of distance between centroids } l \text{ for equilibrium solutions with equal vortices; } - - - \text{ vortices (}\alpha = 0.001\text{); } \cdots \cdots \text{ nonuniform vortices (}\alpha = 1\text{); } \cdots \cdots \text{ nonuniform vortices (}\alpha = 2.25\text{); } \circ \text{, points of exchange of stability.}
\end{align*}
\]
loss of stability and in our case this is what we approximately find (see Fig. 3). We did not attempt a complete convergence analysis. The algorithm used to solve the nonlinear system (up to 700 equations) is rather time-consuming, the Newton-type iterations necessitating an update of the numerically computed Jacobian matrix during the continuation procedure. Furthermore, once a solution converged, a QR algorithm is used to determine the spectrum of the Jacobian matrix for the stability analysis.

The dotted line and broken line of Fig. 3 correspond to the more general case of nonuniform vortices, respectively, with $\alpha=1$ and $\alpha=2.25$. For the first (respectively, the second) case the vorticity at the boundary of a vortex is $1/e$ (respectively $1/10$) of the maximum vorticity at the center. Whereas the qualitative feature is basically recovered for nonuniform computations, the centroids can get closer together before the equilibrium configurations lose their stability and the corresponding $\delta$-values for instability decrease as well. The streamlines inside the pair of nonuniform vortices with $\alpha=2.25$ are depicted in Fig. 4. The solution corresponds to the point of exchange of stability marked as a circle on the broken line of Fig. 3. Whereas the inner streamlines are almost elliptical the outer ones are deformed and again a cusp forms at the outer vortex boundary for this vanishing $\delta$-value.

2. Two unequal vortices

A family of unequal vortex patches, the area of each vortex being equal to its circulation, has been previously considered in Dritschel.18 In order to recover asymptotically this uniform case we again choose $\alpha=0.001$, and set $\alpha_r=1$ in (7) (i.e., $\alpha_r=A_2\Gamma_2$). An example with area ratio of $A_2/A_1=0.5$ (in that case the parameter $\alpha_r=\sqrt{2}$ in (7)) is depicted in Fig. 5. This equilibrium state corresponds to the circle marked on the solid line of Fig. 6, which is the point of exchange of stability. (Again the distances have been normalized, the total circulation being set to $\pi$.) In our computations the zero eigenvalue appears approximately at $l=2.31$ with $\delta=0.51$, which is close to the results depicted in Fig. 4 of Dritschel.18 Nonuniform computations with $\alpha=2.25$ are shown as the broken line in Fig. 6. Again, for this nonuniform case the point of exchange of stability appears for smaller distances between the centroids. For this solution family with $A_2/A_1=0.5$ the streamlines for a stable equilibrium state are shown in Fig. 7(a), whereas the equilibrium state corresponding to the star (respectively, the cross) marked on the broken line in Fig. 6 are depicted in Fig. 7(b) [respectively, Fig. 7(c)]. A rather high resolution has been used for these computations with 12 vorticity levels for each vortex and $m_1=24$ as well as $m_2=54$ for half of each contour, respectively, in vortex 1 and vortex 2. The solutions in Figs. 7(b) and 7(c) are located in the very vicinity of the point of exchange of stability in the parameter space. The main difference between both equilibrium states is the appearance of a cusp (that is a flow stagnation point in the rotating frame of reference) at the outer contour of the smaller vortex, when the solution goes through the marginally stable point. For the equilibrium with a (normalized) gap $\delta=0.26$ [this is approximately the state depicted in Fig. 7(c)], the stream function in the rotating frame of reference for the
The entire flow field has been computed, in a rectangular grid with 100 grid points in both x- and y-direction. This is numerically performed using Green’s function integrals (1).

The streamlines (Fig. 8) exhibit a flow stagnation point. Comparison between Fig. 7 and Fig. 8 indicates that this stagnation point is located at the small vortex boundary leading to a cusp formation, similar to what has been shown in the uniform case (Overman et al. 26).

Steady inviscid fluid motion is characterized by \( \omega = g(\psi) \) for the vorticity \( \omega \) and the stream function \( \psi \). In the present investigation each member of a family is an equilibrium state in a rotating frame of reference, depending on the distance between the vortices. While vorticity remains constant on each streamline, the functional relationship \( \omega = g(\psi) \) in the rotating frame of reference is not unique as shown in Figs. 9(a) and 9(b). In these figures solutions for two unequal vortices with \( A_2/A_1 = 0.5 \) (and \( \alpha = 2.25, \alpha_\omega = 1 \) in (6), (7)) has been considered, and \( \psi \) has been drawn as a function of \( \omega \) for each vortex with gap \( \delta = 1.23, 0.56, 0.28 \).

There is evidence that the stream function inside each vorticity region is fixed as function of the constant vorticity [ranging here from \( 1/\beta \) to \( e^{-\alpha}/\beta \), see (6), (7)] only up to an additive constant depending on the member of the family.
3. Angular impulse and energy

Fixing $\alpha=2.25$ and $\alpha_w=1$ in (6), (7), the angular impulse $J$ and the excess energy $E$ has been computed for equal and unequal vortices. Again, the quantities have been normalized such that the total area and circulation is equal to $\pi$. Once an equilibrium state determined, the normalized angular impulse is, using the scale factor $s=\sqrt{(\Gamma_1+\Gamma_2)/\pi}$,

$$J = \frac{1}{s^2} \int \int \omega(r)|r|^2 dS,$$

whereas the normalized excess energy $E$ is defined by

$$E = \frac{1}{2s^4} \int \int \omega(r)\psi(r) dS'$$

with the stream function

$$\psi(r) = \frac{1}{4\pi} \int \int \log\left(\frac{|r-r'|^2}{s^2}\right) \omega(r') dS'.$$

The above integrals are again expressed inside the vortex regions as vorticity integrals and are hence easily computed as a by-product of our solution procedure. [Note that, owing to the logarithm, the normalization for the stream function adds a constant equal to $\log(s^2)/(\Gamma_1+\Gamma_2)/4\pi]$. The angular impulse for $A_2/A_1=1, 0.75, 0.5$ is depicted in Fig. 10, whereas the excess energy is shown in Fig. 11, as a function of the gap $\delta$. For both families with unequal vortices $J$ exhibits a local minimum where the energy $E$ goes through a local maximum. For the $A_2/A_1=0.5$ solution family this occurs at $\delta=0.25$ close to the point of exchange of stability (cf. Fig. 6). Stability computations for the case with $A_2/A_1=0.75$ give a value $\delta=0.18$ for exchange of stability, which again fits quite well with the local minimum of $J$ and the local maximum of $E$. For equal vortices, the exchange of stability occurs for vanishing $\delta$-values (see Fig. 3) and quantities $J$ and $E$ do not exhibit a local minimum and maximum but reach an optimum for $\delta=0$. For the equilibrium states considered, in agreement with the theory of Saffman, exchange of stability coincides with the minimum of $J$ and the maximum of $E$.

B. Interaction between a point vortex and a finite vortex

Using large values of $\alpha_w$ the limiting case of the interaction between a finite vortex and a point vortex may be approximated. In the following $\alpha=2.25$ has been chosen, and the area $A_2$ has been set to $\pi/100$. The second vortex is now characterized by the circulation $\Gamma_2=\alpha_w\pi/100$ and negligible area. One equilibrium configuration with a strong point vortex for a circulation ratio of $\Gamma_1/\Gamma_2=0.5$ is shown in Fig. 12. The point vortex will hardly undergo any deformation, which justifies to use only 4 points for half of each of the 8 contours (here only the boundary is depicted). For the vortex with a finite region, each of the 10 streamlines has been discretized using 60 points for each half a contour.

We want to compare the centroid distance and the rotation rate at the point of exchange of stability for a finite
vortex destabilized by a point vortex to the results obtained when two finite vortices interact. The interaction between two finite vortices with circulation and areas $\Gamma_1=\Gamma_2=A_2$ is characterized by the circulation ratio $\gamma = \min(\Gamma_2/\Gamma_1, \Gamma_1/\Gamma_2)$. When a point vortex and a finite vortex interact, two cases have to be considered: the vortex with finite region may have higher circulation, or on the contrary the point vortex may be the stronger one. Three cases have been considered: $\gamma=1$ (the case of equal circulation), $\gamma=0.75$ as well as $\gamma=0.5$. The circles depicted in Fig. 13 correspond to the configuration with finite vortices previously studied. The stars on the same figure indicate the points of exchange of stability in the case for which the point vortex has smaller circulation, whereas the crosses are results for a stronger point vortex. To allow a comparison between the different configurations, the distance $l$ has been again normalized multiplying by the scaling factor $\sqrt{\pi/(\Gamma_1+\Gamma_2)}$.

Inspecting Fig. 13, one observes that computations with a stronger point vortex give rise to critical $l$-values close to the results for the finite regions case, for decreasing $\gamma$-values. On the contrary, when the vortex with a finite region is the one with higher circulation, the results diverge.

A basic characteristic of the equilibrium configurations is the system rotation rate $\Omega$. These values for the different marginally stable cases are depicted in Fig. 14. Using a point vortex with higher circulation, the overall rotation rate is again close to the general case, in which two vortices with finite areas are considered. This may provide a way to reduce the general criterion for two vortices merging to the simpler problem of a finite vortex destabilized by the presence of a point vortex.

V. CONCLUDING REMARKS

In this work, we have developed a numerical solution procedure capable of computing inviscid equilibria of vortexes with a nonuniform vorticity distribution. The algorithm can a priori tackle general vortex profiles. We only considered the case of a vortex pair solution of Lamb-type profiles. This study is worth doing since vortices in two-dimensional turbulence are similar to Lamb-type vortices as shown in Jimenez et al.\textsuperscript{3} No dramatic changes from the classical patch vortex case was found when nonuniformity is present. However, the critical centroid distance for instability onset is shown to decrease compared to the patch vortex case. For the nonuniform case considered, with a vorticity ratio $\omega_{\min}/\omega_{\max}=1/10$, the point of marginal stability seems to coincide with the formation of a cusp at the outer vortex boundary. Finally, a relationship is exhibited between the instability threshold for a pair of finite vortices and that of a finite vortex and an equivalent point vortex.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure13}
\caption{Normalized distance between centroids $l$ at the points of exchange of stability, as a function of circulation ratio $\gamma=\min(\Gamma_2/\Gamma_1, \Gamma_1/\Gamma_2)$; $\bigcirc$, finite vortices case; $+$, finite vortex with $\Gamma_1$ and point vortex with $\Gamma_2\leq\Gamma_1$ ($\gamma = \Gamma_2/\Gamma_1$); $\ast$, finite vortex with $\Gamma_1$ and point vortex with $\Gamma_2>\Gamma_1$ ($\gamma = \Gamma_1/\Gamma_2$) ($\alpha=2.25$).}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure14}
\caption{Rotation rate $\Omega$ of marginally stable states, as function of circulation ratio $\gamma$; $\bigcirc$, finite vortices case; $+$, finite vortex with $\Gamma_1$ and point vortex with $\Gamma_2\leq\Gamma_1$ ($\gamma = \Gamma_2/\Gamma_1$); $\ast$, finite vortex with $\Gamma_1$ and point vortex with $\Gamma_2>\Gamma_1$ ($\gamma = \Gamma_1/\Gamma_2$) ($\alpha=2.25$).}
\end{figure}


