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Lectures on Dynamics of Combustion Waves in Premixed Gases

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Lecture IV Hydrodynamic instability of flames

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Lecture 4 : **Hydrodynamic instability of flames**

- 4-1. Jump across an hydrodynamic discontinuity
- 4-2. Linearized Euler equations of an incompressible fluid
- 4-3. Conditions at the front
- 4-4. Dynamics of passive interfaces
- 4-5. Darrieus-Landau instability
- 4-6. Curvature effect: a simplified approach

IV – 1) Jump across an hydrodynamic discontinuity

flame considered as a discontinuity
flame thickness and curvature neglected

$$\Lambda \gg d_L$$

flame $\approx$ surface of zero thickness separating two incompressible flows

Low Mach nb approx + inviscid approx: Euler eqs

$$\partial \rho / \partial t = -\nabla \cdot (\rho \mathbf{u}) \quad \rho D\mathbf{u}/Dt = -\nabla p \Leftrightarrow \partial(\rho \mathbf{u})/\partial t = -\nabla \cdot (p \mathbf{I} + \rho \mathbf{u}\mathbf{u})$$

tilted planar front

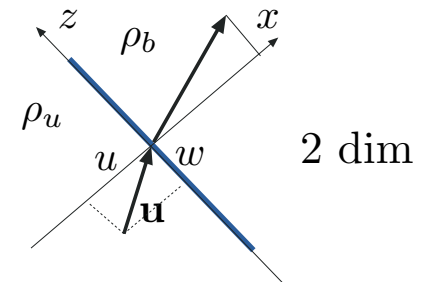
reference frame of the flame

$$\mathbf{r} = (x, z), \quad \mathbf{u} = (u, w)$$

$$\partial \rho / \partial t = -\partial(\rho u) / \partial x - \partial(\rho w) / \partial z,$$

$$\partial(\rho u) / \partial t = -\partial(p + \rho u^2) / \partial x - \partial(\rho u w) / \partial z,$$

$$\partial(\rho w) / \partial t = -\partial(\rho u w) / \partial x - \partial(p + \rho w^2) / \partial z$$



$$\lim_{d_L \rightarrow 0} \int_{d_L} a(x, y, t) dx = 0$$

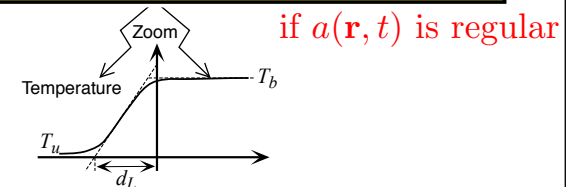
jump relations (reference frame of the flame)

$$[\rho u]_{-}^{+} = 0$$

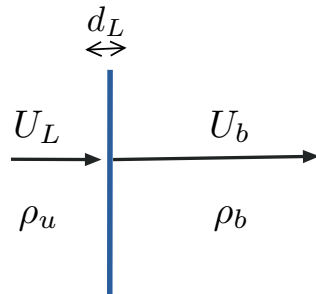
$$\rho u = \rho_u U_L = \rho_b U_b$$

$$[p + \rho u^2]_{-}^{+} = 0$$

$$\rho u \neq 0 \Rightarrow [w]_{-}^{+} = 0$$

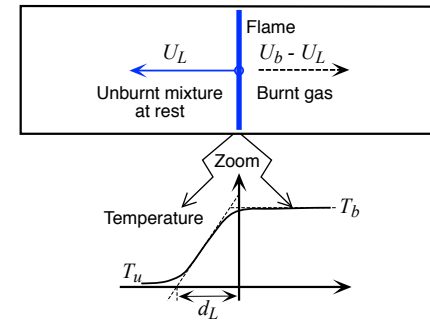


$$\rho_u > \rho_b$$

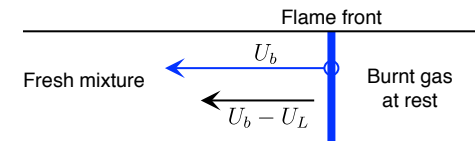


reference frame of the flame front

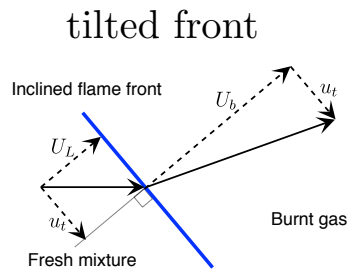
$$\frac{U_b}{U_L} = \frac{\rho_u}{\rho_b} = \frac{T_b}{T_u}$$



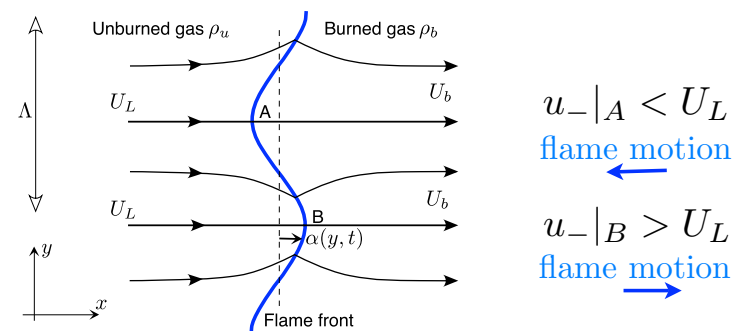
Piston effect



”instantaneous” modification of the flow field, both upstream and downstream
(low Mach nb approx: the speed of sound is infinite, $a \approx \infty$)



deviation of the stream lines

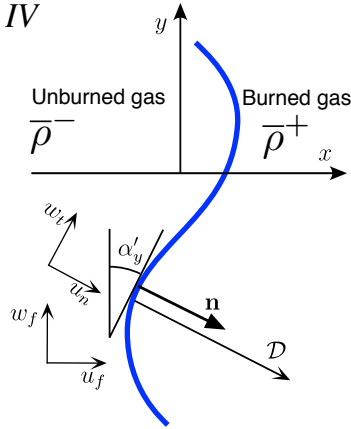


instability mechanism

$$\sigma \propto U_L / \Lambda$$

$$\Lambda \gg d_L, \quad d_L / \Lambda \rightarrow 0$$

P.Clavin IV



equation of the perturbed front $x = \alpha(y, t)$ (reference frame !)
 flow velocity at the front $\mathbf{u}_f = (u_f, w_f)$

$$\mathbf{n}_f = \left(\frac{1}{\sqrt{1 + \alpha_y'^2}}, -\frac{\alpha_y'}{\sqrt{1 + \alpha_y'^2}} \right), \quad u_n \equiv \mathbf{u}_f \cdot \mathbf{n}_f = (u_f - \alpha_y' w_f) / \sqrt{1 + \alpha_y'^2}$$

$$w_{tg} = (w_f + \alpha_y' u_f) / \sqrt{1 + \alpha_y'^2}$$

$$\alpha_y' \equiv \partial \alpha / \partial y$$

$$\dot{\alpha}_t \equiv \partial \alpha / \partial t$$

normal velocity of the front

$$\mathcal{D}_f = \frac{\dot{\alpha}_t}{\sqrt{1 + \alpha_y'^2}}$$

reference frame of the planar unperturbed flame

flow velocity relative to the perturbed front

$$U_n \equiv u_n - \mathcal{D}_f = (u_f - \dot{\alpha}_t - \alpha_y' w_f) / \sqrt{1 + \alpha_y'^2}$$

$$W_{tg} = w_{tg}$$

normal component

tangential component

conservation of mass

$$\bar{\rho}^- U_n^- = \bar{\rho}^+ U_n^+$$

$$\bar{\rho}^- \left(u_f^- - \dot{\alpha}_t - \alpha_y' w_f^- \right) = \bar{\rho}^+ \left(u_f^+ - \dot{\alpha}_t - \alpha_y' w_f^+ \right)$$

conservation of momentum

$$[p + \bar{\rho} U_n^2]_-^+ = 0$$

$$[W_{tg}]_-^+ = 0$$

$$p_f^- + \bar{\rho}^- \frac{\left(u_f^- - \dot{\alpha}_t - \alpha_y' w_f^- \right)^2}{1 + \alpha_y'^2} = p_f^+ + \bar{\rho}^+ \frac{\left(u_f^+ - \dot{\alpha}_t - \alpha_y' w_f^+ \right)^2}{1 + \alpha_y'^2} \quad \left(w_f^- + \alpha_y' u_f^- \right) = \left(w_f^+ + \alpha_y' u_f^+ \right)$$

IV – 2) Linearised Euler equations of an incompressible fluid

$$\begin{aligned} a &= \bar{a} + \delta a \\ \bar{m}_f &= \bar{\rho}^- \bar{u}_f^- = \bar{\rho}^+ \bar{u}_f^+ \end{aligned} \quad \begin{aligned} \frac{\partial}{\partial x} \delta u^\pm + \frac{\partial}{\partial \mathbf{y}} \delta \mathbf{w}^\pm &= 0, \\ \left(\bar{\rho}^\pm \frac{\partial}{\partial t} + \bar{m}_f \frac{\partial}{\partial x} \right) \delta u^\pm &= -\frac{\partial}{\partial x} \delta \pi^\pm \\ \left(\bar{\rho}^\pm \frac{\partial}{\partial t} + \bar{m}_f \frac{\partial}{\partial x} \right) \delta \mathbf{w}^\pm &= -\frac{\partial}{\partial \mathbf{y}} \delta \pi^\pm, \end{aligned} \quad \pi^\pm \equiv p^\pm - \bar{\rho}^\pm g(t)x,$$

$x \rightarrow +\infty$: disturbances remain finite,

$x \rightarrow -\infty$: no disturbances, $\delta u^- = 0$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \delta \pi^\pm = 0$$

transverse coordinates

$$\delta a(x, \mathbf{y}, t) = \tilde{a}(x, t) e^{i\mathbf{k} \cdot \mathbf{y}} \quad \alpha(\mathbf{y}, t) = \tilde{\alpha}(t) e^{i\mathbf{k} \cdot \mathbf{y}}$$

pressure

$$\frac{\partial^2 \tilde{\pi}^\pm}{\partial x^2} - |\mathbf{k}|^2 \tilde{\pi}^\pm = 0$$

$$\tilde{\pi}^\pm(x, t) = \tilde{\pi}_f^\pm(t) e^{\mp |\mathbf{k}| x}$$

flow velocity

$$\frac{\partial \tilde{u}^\pm}{\partial x} + i\mathbf{k} \cdot \tilde{\mathbf{w}}^\pm = 0 \quad \bar{\rho}^\pm \left(\frac{\partial}{\partial t} + \bar{u}^\pm \frac{\partial}{\partial x} \right) \tilde{u}^\pm(x, t) = \pm |\mathbf{k}| \tilde{\pi}_f^\pm(t) e^{\mp |\mathbf{k}| x}$$

general solution to the homogeneous equation + particular solution

$$\tilde{u}^\pm(x, t) = \tilde{u}_R^\pm(x, t) + \tilde{u}_P^\pm(x, t)$$

$$\frac{\partial \tilde{u}_R^\pm}{\partial t} + \bar{u}^\pm \frac{\partial \tilde{u}_R^\pm}{\partial x} = 0, \quad \bar{\rho}^\pm \left(\frac{d}{dt} \mp \bar{u}^\pm k \right) \tilde{u}_P^\pm(t) = \pm k \tilde{\pi}_f^\pm(t)$$

$$\tilde{u}^\pm(x, t) = \tilde{u}_R^\pm(x, t) + \tilde{u}_P^\pm(x, t)$$

$$k \equiv |\mathbf{k}| = 2\pi/\Lambda \quad \tilde{u}^- = \tilde{u}_P^- = \tilde{u}_f^-(t)e^{kx} \quad \tilde{u}_P^+ = \tilde{u}_p^+(t)e^{-kx}$$

potential flows

$$\frac{\partial \tilde{u}_R^\pm}{\partial t} + \bar{u}^\pm \frac{\partial \tilde{u}_R^\pm}{\partial x} = 0, \quad \tilde{u}_R^\pm = \tilde{u}_r^\pm(t - x/\bar{u}^\pm), \quad \tilde{u}_R^- = 0, \quad \tilde{u}_R^+ = \tilde{u}_r^+(t - x/\bar{u}^+),$$

vorticity of the burnt gas flow

3 unknown functions: $\tilde{u}_f^-(t)$, $\tilde{u}_p^+(t)$, $\tilde{u}_r^+(t)$

$$x < 0 : \quad \begin{cases} \tilde{u}^-(x, t) = \tilde{u}_f^-(t)e^{+kx}, \\ k\tilde{\pi}^-(x, t) = -\bar{\rho}^- \left(\frac{d}{dt} + \bar{u}^- k \right) \tilde{u}_f^-(t)e^{+kx}, \end{cases}$$

$$x > 0 : \quad \begin{cases} \tilde{u}^+(x, t) = \tilde{u}_p^+(t)e^{-kx} + \tilde{u}_r^+(t - x/\bar{u}^+), \\ k\tilde{\pi}^+(x, t) = \bar{\rho}^+ \left(\frac{d}{dt} - \bar{u}^+ k \right) \tilde{u}_p^+(t)e^{-kx}, \end{cases}$$

$$\mathbf{i}\mathbf{k} \cdot \tilde{\mathbf{w}}^-(x, t) = -k\tilde{u}^-(x, t), \quad \mathbf{i}\mathbf{k} \cdot \tilde{\mathbf{w}}^+(x, t) = -\frac{\partial}{\partial x} \tilde{u}^+(x, t).$$

4 boundary conditions at the flame front involving the additional unknown $\tilde{\alpha}(t)$

2 for the conservation of mass (inner flame structure not modified)

$$\delta m_f^- = \delta m_f^+ = 0$$

$$m \equiv \rho(u - \partial\alpha/\partial t)$$

2 for the conservation of normal and tangential momentum

IV-3) Conditions at the front

notation

$$a(x, t) \\ a_f(t) \equiv a(x = 0, t)$$

Mass

$$\boxed{\bar{\rho}^- (\delta u_f^- - \dot{\alpha}_t) = \bar{\rho}^+ (\delta u_f^+ - \dot{\alpha}_t) = 0} \Rightarrow \delta u_f^- = \delta u_f^+ = \dot{\alpha}_t \quad \tilde{u}_f^-(t) = \tilde{u}_p^+(t) + \tilde{u}_r^+(t) = d\tilde{\alpha}/dt$$

 $\Downarrow$

$$\frac{d\tilde{u}_r^+}{dt} = -\frac{d\tilde{u}_p^+}{dt} + \frac{d^2\tilde{\alpha}}{dt^2}$$

Tangential momentum

$$\boxed{\frac{\partial}{\partial y} (w_f^- + \alpha'_y \bar{u}^-) = \frac{\partial}{\partial y} (w_f^+ + \alpha'_y \bar{u}^+)} \Rightarrow k\tilde{u}_p^+(t) + \frac{1}{\bar{u}^+} \frac{d\tilde{u}_r^+(t)}{dt} + k\tilde{u}_f^-(t) = \bar{m}_f \left(\frac{1}{\bar{\rho}^+} - \frac{1}{\bar{\rho}^-} \right) k^2 \tilde{\alpha}(t)$$

elimination of $d\tilde{u}_r^+/dt$

$$ik \cdot \tilde{\mathbf{w}}^-(x, t) = -k\tilde{u}^-(x, t), \quad ik \cdot \tilde{\mathbf{w}}^+(x, t) = -\frac{\partial}{\partial x} \tilde{u}^+(x, t). \\ \partial \alpha'_y / \partial y \rightarrow -k^2 \tilde{\alpha}(t)$$

 $\Downarrow$

$$\left(\frac{1}{\bar{u}^+} \frac{d}{dt} - k \right) \tilde{u}_p^+ = -\bar{m}_f \left(\frac{1}{\bar{\rho}^+} - \frac{1}{\bar{\rho}^-} \right) k^2 \tilde{\alpha} + \frac{1}{\bar{u}^+} \frac{d^2\tilde{\alpha}}{dt^2} + k\tilde{u}_f^-$$

 $\uparrow k \frac{d\tilde{\alpha}}{dt}$

Normal momentum

$$\boxed{\delta p_f^- + 2\bar{\rho}^- \bar{u}_f^- (\delta u_f^- - \dot{\alpha}_t) = \delta p_f^+ + 2\bar{\rho}^+ \bar{u}_f^+ (\delta u_f^+ - \dot{\alpha}_t)} \Rightarrow \tilde{\pi}_f^+ - \tilde{\pi}_f^- = (\bar{\rho}^- - \bar{\rho}^+) g(t) \tilde{\alpha}(t)$$

hydrostatic pressure

 $\Downarrow$

$$x < 0 : \begin{cases} \tilde{u}^-(x, t) = \tilde{u}_f^-(t) e^{+kx}, \\ k\tilde{\pi}^-(x, t) = -\bar{\rho}^- \left(\frac{d}{dt} + \bar{u}^- k \right) \tilde{u}_f^-(t) e^{+kx}, \end{cases} \Rightarrow$$

$$x > 0 : \begin{cases} \tilde{u}^+(x, t) = \tilde{u}_p^+(t) e^{-kx} + \tilde{u}_r^+(t - x/\bar{u}^+), \\ k\tilde{\pi}^+(x, t) = \bar{\rho}^+ \left(\frac{d}{dt} - \bar{u}^+ k \right) \tilde{u}_p^+(t) e^{-kx}, \end{cases}$$

$$\bar{m}_f \left(\frac{1}{\bar{u}^+} \frac{d}{dt} - k \right) \tilde{u}_p^+ + \bar{m}_f \left(\frac{1}{\bar{u}^-} \frac{d}{dt} + k \right) \tilde{u}_f^- = (\bar{\rho}^- - \bar{\rho}^+) k g(t) \tilde{\alpha}(t)$$

 $\uparrow \frac{d\tilde{\alpha}}{dt}$

Equation for the front

elimination of $\tilde{u}_p^+$ $\Downarrow$

$$\tilde{u}_f^- = d\tilde{\alpha}/dt \Rightarrow (\bar{\rho}^- + \bar{\rho}^+) \frac{d^2\tilde{\alpha}}{dt^2} + 2\bar{m}_f k \frac{d\tilde{\alpha}}{dt} - k[(\bar{\rho}^- - \bar{\rho}^+) g(t) + (\bar{u}^+ - \bar{u}^-) \bar{m}_f k] \tilde{\alpha} = 0$$

IV-4) Dynamics of a passive interface



Rayleigh

$$\overline{m}_f = 0$$

$$(\bar{\rho}^- + \bar{\rho}^+) \frac{d^2 \tilde{\alpha}}{dt^2} + \cancel{2\bar{m}_f k} \frac{d\tilde{\alpha}}{dt} - k[(\bar{\rho}^- - \bar{\rho}^+)g(t) + (\bar{u}^+ - \bar{u}^-) \cancel{\bar{m}_f k}] \tilde{\alpha} = 0$$

Fourier mode

$$\tilde{\alpha}(t) = \hat{\alpha} e^{\sigma t}$$

$$\alpha(\mathbf{y}, t) = \tilde{\alpha}(t) e^{i\mathbf{k} \cdot \mathbf{y}}$$

$\text{Re}(\sigma) < 0$: linearly stable

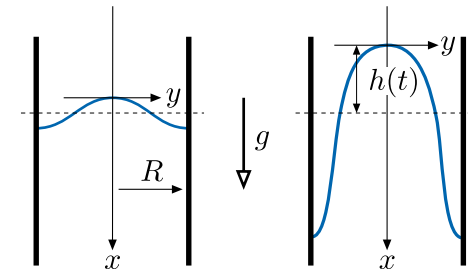
$\text{Re}(\sigma) > 0$: linearly unstable

Rayleigh-Taylor instability

$$g = \text{cst.} \quad (\bar{\rho}^- - \bar{\rho}^+)g > 0$$

$$\sigma^2 - A_t k g = 0, \quad A_t > 0$$

Rayleigh-Taylor bubble (upwards propagation)



$$g > 0, \quad A_t \equiv \frac{\rho_- - \rho_+}{\rho_- + \rho_+} > 0 \quad \sigma = \sqrt{A_t g k} \quad U_{\text{bubble}} = 0.361 \sqrt{2gR}$$

ok with dimension

Gravity waves

$$g = \text{cst.} \quad (\bar{\rho}^- - \bar{\rho}^+)g < 0$$

$$\varpi \equiv \text{Im } \sigma \neq 0$$

$$\tilde{\alpha}(t) = \hat{\alpha} e^{i\varpi t} \quad \varpi = B \sqrt{gk}$$

$$B \equiv \sqrt{\frac{(\rho_+ - \rho_-)}{(\rho_+ + \rho_-)}}$$

Faraday (parametric) instability. Mathieu's equation

$$g(t) \text{ oscillating} \quad \frac{d^2 \tilde{\alpha}}{dt^2} + \varpi_o^2 [1 + \epsilon \cos(\varpi \tau)] \tilde{\alpha} = 0$$



Landau

IV-5) Darrieus-Landau instability of flames

Darrieus 1938
Landau 1944

$$g = 0 \quad \bar{m}_f = \bar{\rho}^- \bar{u}^- = \bar{\rho}^+ \bar{u}^+$$

$$(\bar{\rho}^- + \bar{\rho}^+) \frac{d^2 \tilde{\alpha}}{dt^2} + 2\bar{m}_f k \frac{d\tilde{\alpha}}{dt} - (\bar{u}^+ - \bar{u}^-) \bar{m}_f k^2 \tilde{\alpha} = 0$$

$$\tilde{\alpha}(t) = \hat{\alpha} e^{\sigma t}$$

$$v_b \equiv \bar{\rho}^- / \bar{\rho}^+ = \bar{u}^+ / \bar{u}^- > 1 \quad (\bar{\rho}^- + \bar{\rho}^+) \sigma^2 + 2\bar{m}_f k \sigma - (\bar{u}^+ - \bar{u}^-) \bar{m}_f k^2 = 0$$

$$\bar{u}_- \equiv U_L \quad \frac{\sigma}{U_L k} = \frac{1}{1 + v_b^{-1}} \left[-1 \pm \sqrt{1 + v_b - v_b^{-1}} \right]$$

$$\sigma = \mathcal{A} U_L k, \quad \mathcal{A} > 0$$

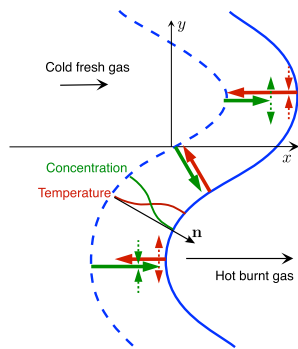
$d_L / \Lambda \rightarrow 0$: no length scale in the problem; dimensional analysis $\Rightarrow \sigma \propto U_L k$

$$\rho_u \gg \rho_b : \sigma = \sqrt{U_b U_L} k$$

$$(\rho_u - \rho_b) / \rho_u \ll 1 : \sigma = (U_b - U_L) k / 2$$

$k = 2\pi / \Lambda$ shorter is the wavelength stronger is the instability !?

however the analysis is valid only in the limit $d_L / \Lambda \rightarrow 0$

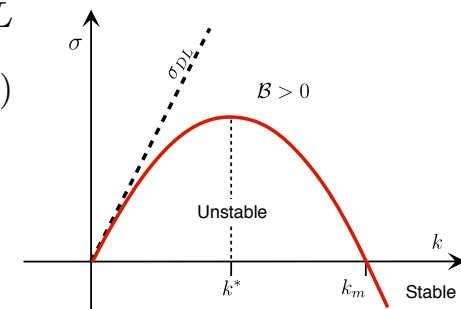


Stabilisation at small wavelength, $\Lambda \approx d_L$

$$\frac{\partial \alpha}{\partial t} = \mathcal{B} D_T \frac{\partial^2 \alpha}{\partial y^2}, \quad \sigma_{diff} \equiv 1 / \tau_{diff} = -\mathcal{B} D_T k^2 = -\mathcal{B} U_L k (d_L k)$$

first order correction $\mathcal{B} > 0$?

$$k d_L < 1 : \sigma = \mathcal{A} U_L k - \mathcal{B} k^2 d_L + \dots$$



VI-6) Curvature effect: a simplified approach

modification to the inner flame structure $\delta m_f^- = \delta m_f^+ \neq 0$

first order in perturbation analysis $d_L/\Lambda \ll 1$ $\delta m_f^-/\bar{\rho}^- \equiv (\delta u_f^- - \dot{\alpha}_t) = -\mathcal{B}D_T\partial^2\alpha/\partial y^2$
 $\tilde{m}_f^-(t)/\bar{m}_f \approx \mathcal{B}d_Lk^2\tilde{\alpha}(t) \quad D_T = U_Ld_L$

Normal momentum $\delta p_f^- + 2\bar{\rho}^- \bar{u}_f^- (\delta u_f^- - \dot{\alpha}_t) = \delta p_f^+ + 2\bar{\rho}^+ \bar{u}_f^+ (\delta u_f^+ - \dot{\alpha}_t)$

(flame notations: $\bar{\rho}^+ \rightarrow \rho_b$, $\bar{\rho}^- \rightarrow \rho_u$, $\rho_u > \rho_b$) $\tilde{\pi}_{f+} - \tilde{\pi}_{f-} = -2\bar{m}_f \left(\frac{1}{\rho_b} - \frac{1}{\rho_u} \right) \tilde{m}_f(t) + (\rho_u - \rho_b)g(t)\tilde{\alpha}(t)$

equation for the flame front (correction due to curvature, finite thickness effect $kd_L \neq 0$)

$$(\rho_u + \rho_b) \frac{d^2 \tilde{\alpha}}{dt^2} + 2\bar{m}_f k \frac{d\tilde{\alpha}}{dt} (1 + \cancel{\mathcal{B}kd_L}) = k\tilde{\alpha}(\rho_u - \rho_b) [g(t) + U_b U_L k (1 - 2\mathcal{B}kd_L)]$$

flame propagating downwards $g < 0$

$$\frac{1}{k_m} \equiv 2\mathcal{B}d_L$$

$$\left(1 + \frac{\rho_b}{\rho_u}\right) \frac{d^2 \tilde{\alpha}}{dt^2} + 2U_L k \frac{d\tilde{\alpha}}{dt} = \left(\frac{\rho_u}{\rho_b} - 1\right) k \left[-\frac{\rho_b}{\rho_u} |g| + U_L^2 k \left(1 - \frac{k}{k_m}\right) \right] \tilde{\alpha}$$

non-dimensional parameters

$$v_b \equiv \bar{\rho}^-/\bar{\rho}^+ = \bar{u}^+/\bar{u}^- > 1$$

$$s = \sigma\tau_L \quad \kappa \equiv kd_L \quad \kappa_m \equiv 1/(2\mathcal{B}) \quad \mathcal{G}_0 \equiv (\rho_b/\rho_u)\text{Fr}^{-1} \quad \text{Fr}^{-1} \equiv |g|d_L/U_L^2$$

$$(1 + v_b^{-1})s^2 + 2\kappa s - (v_b - 1)\kappa \left[-\mathcal{G}_0 + \kappa \left(1 - \frac{\kappa}{\kappa_m}\right) \right] = 0$$

Stability limits of flames propagating downwards $\sigma = 0$

marginal wavenumber

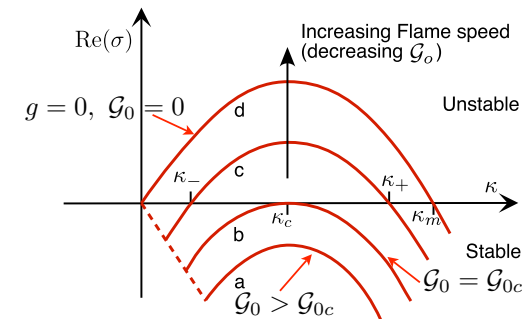
$$\left[-\mathcal{G}_0 + \kappa \left(1 - \frac{\kappa}{\kappa_m}\right) \right] = 0,$$

Stability limits of flames propagating downwards

non-dimensional parameters $\kappa \equiv kd_L$ $\kappa_m \equiv 1/(2\mathcal{B})$ $\mathcal{G}_0 \equiv (\rho_b/\rho_u)\text{Fr}^{-1}$ $\text{Fr}^{-1} \equiv |g|d_L/U_L^2$

$$s = \sigma\tau_L \quad (1 + v_b^{-1})s^2 + 2\kappa s - (v_b - 1)\kappa \left[-\mathcal{G}_o + \kappa \left(1 - \frac{\kappa}{\kappa_m} \right) \right] = 0$$

marginal wavenumber $\sigma = 0$ $\left[-\mathcal{G}_o + \kappa \left(1 - \frac{\kappa}{\kappa_m} \right) \right] = 0,$



gravity stabilizes the large wavelengths of slow propagating flame

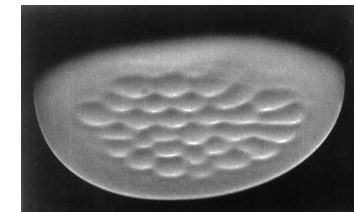
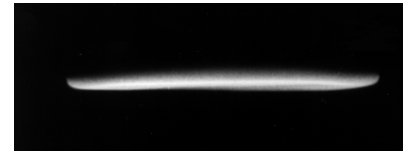
curvature stabilizes the small wavelengths

gravity stabilizes the large wavelengths of slow propagating flame $U_L < 10\text{cm/s}$

instability threshold $U_L \approx 10\text{cm/s}$

$$\mathcal{G}_{oc} = \frac{k_c d_L}{2}, \quad k_c = \frac{\kappa_m}{2}, \quad U_{Lc} = \sqrt{2 \frac{\rho_b}{\rho_u} \frac{|g|}{k_c}}$$

OK with experiments by Boyer Quinard and Searby (1982)



Flames propagating upwards: bubble flames

