Tsinghua-Princeton-CI Summer School
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Lectures on Dynamics of Combustion Waves in Premixed Gases

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Lecture VIII
Thermo-acoustic instabilities. Vibratory flames

Lecture 8: Thermo-acoustic instabilities

Lecture 8-1. Rayleigh criterion

Acoustic waves in a reactive medium
Sound emission by a localized heat source
Linear growth rate

Lecture 8-2. Admittance & transfer function

Flame propagating in a tube
Pressure coupling
Velocity and acceleration coupling

Lecture 8-3. Vibratory instability of flames

Acoustic re-stabilisation and parametric instability (Mathieu's equation)
Flame propagating downward (sensitivity to the Markestein number)
Bunsen flame in an acoustic field

VIII-1) Rayleigh criterion

Acoustic waves in a reactive medium

Lord Rayleigh 1878

Ideal gas

$$p = (c_p - c_v)\rho T \qquad \frac{c_p}{c_p - c_v}\frac{\mathrm{D}p}{\mathrm{D}t} = c_p T \frac{\mathrm{D}\rho}{\mathrm{D}t} + c_p \rho \frac{\mathrm{D}T}{\mathrm{D}t} \qquad \mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \mathbf{u}.\nabla$$

$$a^2 = (c_p/c_v)(c_p - c_v)T \qquad \rho c_p \frac{\mathrm{D}}{\mathrm{D}t}T = \frac{c_p}{c_p - c_v}\frac{\mathrm{D}}{\mathrm{D}t}p - \frac{c_v}{c_p - c_v}a^2\frac{\mathrm{D}}{\mathrm{D}t}\rho$$

Energy conservation

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}$$

$$\frac{c_v}{c_p - c_v}\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{c_v}{c_p - c_v}a^2\frac{\mathrm{D}\rho}{\mathrm{D}t} = \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}$$

$$\boxed{\mathrm{D}p/\mathrm{D}t - a^2\mathrm{D}\rho/\mathrm{D}t = \dot{q}_\gamma}$$

isentropic acoustic
$\delta p = a^2 \delta\rho$

$$\dot{q}_\gamma \equiv (\gamma - 1)\left[\nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}\right] \qquad \gamma \equiv c_p/c_v$$

$\dot{q}_\gamma(\mathbf{r}, t) =$ heat transfer + heat release

(rate of energy transfert per unit volume)

$$\boxed{\mathrm{D}p/\mathrm{D}t - a^2\mathrm{D}\rho/\mathrm{D}t = \dot{q}_\gamma}$$

$$\dot{q}_\gamma \equiv (\gamma - 1)\left[\nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}\right] \qquad \gamma \equiv c_p/c_v$$

Linearization around a uniform state $\quad \nabla\overline{a} \approx 0,$

$$\rho = \overline{\rho} + \rho', \quad p = \overline{p} + p', \quad \mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \quad \dot{q}_\gamma = \overline{\dot{q}_\gamma} + \dot{q}'_\gamma$$

Mean flow velocity neglected in front of the sound speed $\quad \overline{\mathbf{u}}.\nabla \approx 0$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \quad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p \quad \Rightarrow \quad \partial\rho'/\partial t = -\overline{\rho}\nabla.\mathbf{u}', \quad \overline{\rho}\partial\mathbf{u}'/\partial t = -\nabla p',$$

$$\partial^2\rho'/\partial t^2 = \Delta p'$$

Approximations

$a,\ c_p,\ c_v;$ constant

$$\partial/\partial t \qquad \partial p'/\partial t - \overline{a}^2\partial\rho'/\partial t = \dot{q}'_\gamma$$

elimination of ρ' $\quad \boxed{\partial^2 p'/\partial t^2 - \overline{a}^2\Delta p' = \partial\dot{q}'_\gamma/\partial t}$

Sound emission by a localized heat source in free space

classical problem of acoustics

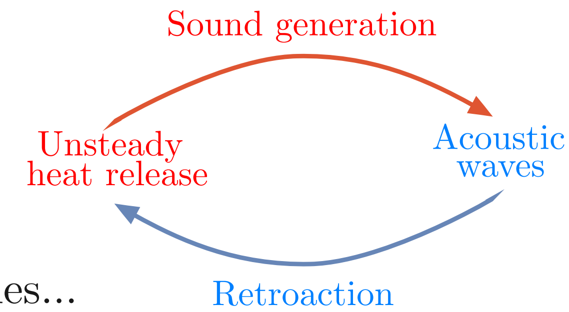
acoustic wavelength $\gg$ size of the combustion zone $\quad \partial\dot{q}'_\gamma(\mathbf{r},t)/\partial t = \delta(\mathbf{r})\ddot{\Omega}(t) \quad \ddot{\Omega}(t) \equiv \partial\dot{\Omega}(t)/\partial t, \quad \boxed{\dot{\Omega}(t) = \iiint \dot{q}'_\gamma(\mathbf{r}',t)\mathrm{d}^3\mathbf{r}'}$

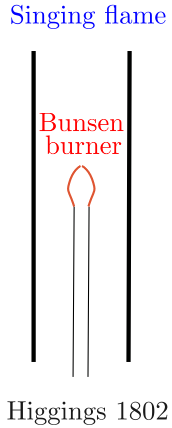
Green's retarded propagator $\quad (1/\overline{a}^2)\partial^2 G/\partial t^2 - \Delta G = \delta(\mathbf{r})\delta(t), \quad G(\mathbf{r},t) = \overline{a}\delta(r - \overline{a}t)/4\pi r$

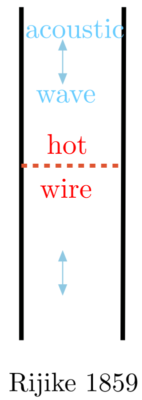
spherical geometry $\quad \boxed{p'(\mathbf{r},t) =} \dfrac{1}{4\pi\overline{a}^2}\iiint \dfrac{1}{r'}\dfrac{\partial}{\partial t}\dot{q}'_\gamma(\mathbf{r}', t - r/\overline{a})\mathrm{d}^3\mathbf{r}' = \boxed{\dfrac{\ddot{\Omega}(t - r/\overline{a})}{4\pi\overline{a}^2 r}}, \qquad r = |\mathbf{r}|$

Liner growth rate

retro-action loop:







Rocket engines, gas turbines...

Simplest retro-action mechanism: pressure coupling + 1-D geometry

$$\boxed{\delta\dot{q}_\gamma = b\,\delta p/\tau_{ins}} \qquad \delta p(x,t) = \sum_{k=-\infty}^{\infty} \tilde{p}_k(t)\mathrm{e}^{\mathrm{i}kx} \qquad k = 2\pi n/L$$

$$\partial^2 p'/\partial t^2 - \bar{a}^2 \Delta p' = \partial \dot{q}'_\gamma/\partial t \qquad \Rightarrow \frac{\mathrm{d}^2\tilde{p}_k}{\mathrm{d}t^2} - \frac{b}{\tau_{ins}}\frac{\mathrm{d}\tilde{p}_k}{\mathrm{d}t} + \bar{a}^2k^2\tilde{p}_k = 0$$

$$\tilde{p}_k = \mathrm{e}^{\sigma t} \qquad 2\sigma\tau_{ins} = b \pm \sqrt{\cancel{b^2} - 4\bar{a}^2k^2\tau_{ins}^2} \qquad \frac{1}{\tau_{ins}} \ll \omega_k = \bar{a}k$$

$$\mathrm{Im}(\sigma) = \omega_k + .., \quad \boxed{\mathrm{Re}(\sigma) = b/(2\tau_{inst}) + ..} \Rightarrow \begin{cases} b > 0 \text{ fluctuations of heat release and pressure } \boxed{\text{in phase: instability}} \\ b < 0 \text{ fluctuations of heat release and pressure } \boxed{\text{out of phase: stability}} \end{cases}$$

More general retro-action mechanism $\boxed{\delta\dot{q}'_\gamma(x,t) = \frac{1}{\tau_{ins}}\int_{-\infty}^{t} b(t-t')\delta p'(x,t')\mathrm{t}'}$

$$b(\tau) = \int_{-\infty}^{+\infty} r(\omega)\mathrm{e}^{\mathrm{i}\omega\tau_d(\omega)}\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega + \mathrm{c.c.} \qquad r(\omega) > 0 \qquad \omega\tau_d(\omega) \text{ is the phase lag}$$

$$\boxed{-\pi/2 < \omega_k\tau_d(\omega_k) < +\pi/2 \ : \ \text{Instability}}$$

Nonlinear study: limit cycles in the unstable case

VIII-2) Admittance & transfer function

laser tomography (Boyer 1980)

Flame propagating in a tube

thickness of the flame brush $\gg$ acoustic wavelength

gas expansion $\Rightarrow$ jump of the fluctuations of the flow velocity (acoustics)

$$(\delta u_b - \delta u_u)/U_L = O(1)$$

acoustic pressure
$\delta p = \rho a \delta u$

jump of the pressure is negligible

$p/\rho a^2 = O(1) \qquad (\delta p_b - \delta p_u)/p = O(U_L/a)$ $\qquad \delta p_f$: fluctuation of the pressure at the flame

averaged energy flux (/period) combustion $\rightarrow$ acoustic $\qquad \boxed{\dot{\mathcal{E}}_t = \overline{(\delta u_b - \delta u_u)\delta p_f}}$

mass conservation (quasi-isobaric combustion) $\quad \nabla.\mathbf{u} = \dfrac{1}{T}\dfrac{\mathrm{D}T}{\mathrm{D}t} = \dfrac{\dot{q}_\gamma/(\gamma-1)}{\rho c_p T} = \dfrac{\dot{q}_\gamma}{\rho a^2} \Rightarrow (\delta u_b - \delta u_u) = \displaystyle\int_{\text{flame brush}} \dfrac{\delta \dot{q}_\gamma}{\rho a^2}\mathrm{d}x$

Pressure coupling

Definition of the admittance function $\quad \mathcal{Z}(\omega)$

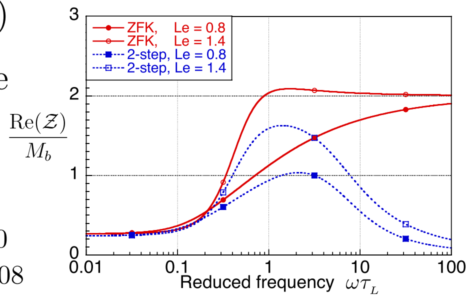
$\delta u(t) = \mathrm{Re}\left[\hat{u}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right] \qquad \delta p(t) = \mathrm{Re}\left[\hat{p}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right] \qquad \boxed{(\hat{u}_b - \hat{u}_u) = \mathcal{Z}(\omega)\,\hat{p}_f/\rho_b a_b} \quad \Leftarrow \quad$ (Rayleigh: $\delta\dot{q}_\gamma$ v.s. δp)

$\dot{\mathcal{E}}_t = \dfrac{1}{4\rho_b a_b}(\mathcal{Z}\hat{p}_f\hat{p}_f^* + \mathcal{Z}^*\hat{p}_f\hat{p}_f^*) = \dfrac{1}{2}[\mathrm{Re}\,\mathcal{Z}(\omega)]\dfrac{|\hat{p}_f|^2}{\rho_b a_b} \qquad \boxed{\text{instability : } \quad \mathrm{Re}(\mathcal{Z}) > 0}$

Analytical study of a planar flame submitted to a fluctuation of pressure $(\beta \to \infty) \qquad \delta T_f/T_f \propto \delta p_f/p_f$

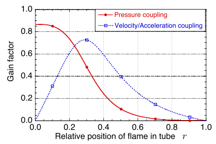
$|\mathcal{Z}| = \mathrm{O}(M_b)$

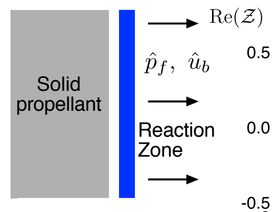
gaseous flame



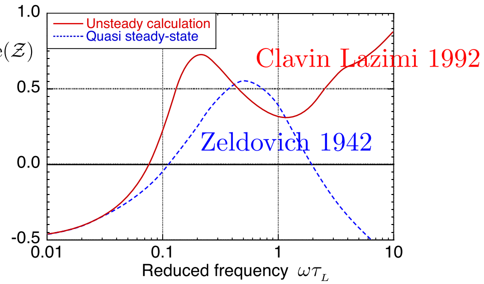
Clavin et al. 1990
Clavin Searby 2008

$\dfrac{\tau_a}{\tau_{ins}} \propto (\gamma-1)M_b\dfrac{E}{k_B T_b}$
coeff depends on
the position in the tube
as δp_f does



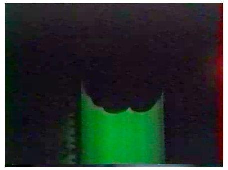


Clavin Lazimi 1992



$\delta u(t) = \mathrm{Re}\left[\hat{u}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right] \qquad \delta p(t) = \mathrm{Re}\left[\hat{p}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right]$

Velocity and acceleration coupling



burned gas

$\uparrow \delta u_b$

$\uparrow \delta u_u$

fresh mixture

fluctuating velocity $\Rightarrow$ modification to flame geometry

$\Rightarrow$ fluctuation of heat release through the flame surface

Transfer function for a flame in a tube $\mathcal{T}_r(\omega)$ $\boxed{(\hat{u}_b - \hat{u}_u) = \mathcal{T}_r(\omega)\hat{u}_u}$ $\quad \dot{\mathcal{E}}_t = (1/4)(\mathcal{T}_r\hat{u}_u\hat{p}_f^* + \mathcal{T}_r^*\hat{u}_u^*\hat{p}_f)$

$\hat{u}\hat{p}_f^* = -\hat{u}^*\hat{p}_f \qquad \dot{\mathcal{E}}_t = \mathrm{Im}\,\mathcal{T}_r(\omega)(\mathrm{i}\hat{u}_u\hat{p}_f^*)/2$

phase quadrature (acoustic mode of a tube) Real number (sign depends on position)

Weakly cellular flame propagating downward in an acoustic wave

acceleration of a curved flame $\Rightarrow$ modulation of the flame surface $S = \int \mathrm{d}y\sqrt{1+\alpha_y'^2}$

$$\int \delta\dot{q}\,\mathrm{d}x = \rho_u U_L c_p\,(T_b - T_u)\,\delta S/S_o \qquad \delta u_b - \delta u_u = \int \frac{\delta\dot{q}}{\rho a^2}\mathrm{d}x \quad \Rightarrow \quad \boxed{\delta u_b - \delta u_u = (T_b/T_u - 1)U_L\delta S/S_o}$$

fluctuation of heat release rate/ cross-section aera

Consider a curved front slightly perturbed

unperturbed perturbation

$$x = \alpha(y,t) \qquad \alpha(y,t) = \tilde{\alpha}(t)\cos(ky) \qquad \tilde{\alpha}(t) = \tilde{\alpha}_0 + \hat{\alpha}_1\mathrm{e}^{\mathrm{i}\omega t} + c.c$$

$k\tilde{\alpha}_0 \ll 1 \qquad |\tilde{\alpha}_1| \ll \tilde{\alpha}_0$ (linear response ok) $\Rightarrow$ $\boxed{\delta S/S_o = (k^2/2)\tilde{\alpha}_o\hat{\alpha}_1\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{c.c.}}$

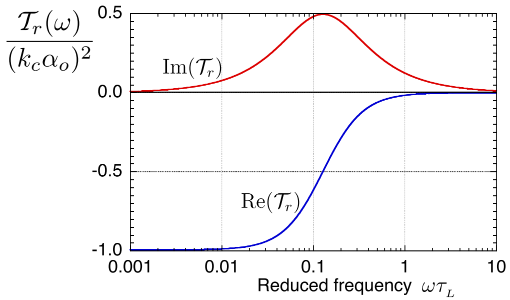
$\tilde{\alpha}_1$ vs $\hat{u}_u$? $\qquad g'(t) = \mathrm{Re}\left[i\omega\,\hat{u}_u\mathrm{e}^{\mathrm{i}\omega t}\right] \qquad \overline{g} > 0$

lecture IV: $$\left(1+\frac{\rho_b}{\rho_u}\right)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2(U_Lk)\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} - \left(\frac{\rho_u}{\rho_b} - 1\right)k\left[-\frac{\rho_b}{\rho_u}[\overline{g} + g'(t)] + U_L^2k\left(1 - \frac{k}{k_m}\right)\right]\tilde{\alpha} = 0$$

Analytical expression $(k = k_c)$

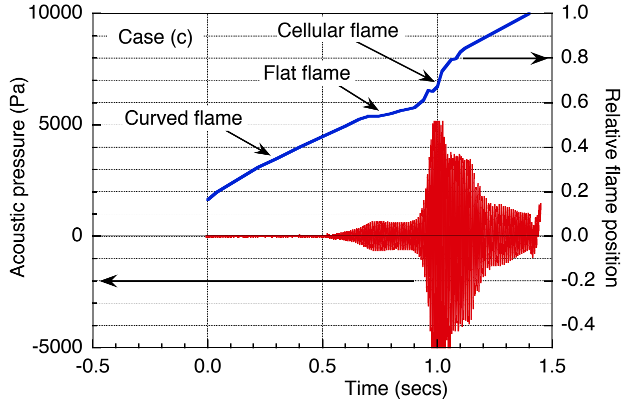
(Pelcé Rochewerger 1992)

ok for the primary instability



VIII-3) Vibratory instability of flames

primary instability + re-stabilisation + parametric instability

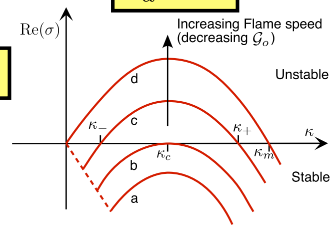


Acoustic re-stabilisation and parametric instability

Markstein 1964

$$\boxed{u_a = 0}$$

$$\tau' \equiv t/\tau_h, \quad \tau_h \equiv 1/(U_L k), \quad \varpi \equiv \omega\tau_h, = (\omega\tau_L)/(kd_L) \qquad \kappa = kd_L$$



$$\boxed{g'(t) = \omega u_a U_L \cos(\omega t)}$$

$$\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}\tau'^2} + 2\mathsf{B}\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}\tau'} + \left[-\mathsf{D} + \varpi^2\mathsf{C}\cos(\varpi\tau')\right]\tilde{\alpha} = 0$$

$$\upsilon_b \equiv \rho_u/\rho_b > 1 \qquad \mathsf{B} \equiv \frac{\upsilon_b}{\upsilon_b + 1} \qquad \mathsf{C} \equiv \left(\frac{\upsilon_b - 1}{\upsilon_b + 1}\right)\frac{u_a}{\varpi},$$

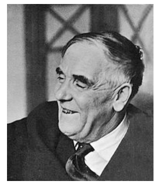
$$\mathsf{D} \equiv \upsilon_b\left(\frac{\upsilon_b - 1}{\upsilon_b + 1}\right)\frac{\mathsf{N}}{\kappa} \qquad \mathsf{N}(\kappa) \equiv -\mathcal{G}_o + \kappa - \kappa^2/\kappa_m \qquad \mathcal{G}_o \equiv \upsilon_b^{-1}|g|d_L/U_L^2$$

Mathieu's equation. Kapitza pendulum

$$\mathrm{t} \equiv \varpi\tau' \qquad Y(\mathrm{t}) \equiv \mathrm{e}^{\mathsf{B}\tau'}\tilde{\alpha}$$

$$\boxed{\frac{\mathrm{d}^2 Y}{\mathrm{dt}^2} + \{\Omega + h\,\cos(\mathrm{t})\}\,Y = 0}$$

$$\Omega = -\frac{(\mathsf{D} + \mathsf{B}^2)}{\varpi^2} \qquad h = \mathsf{C}$$

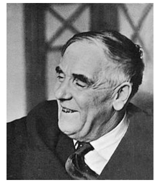


Kapitza 1951

Mathieu's equation. Kapitza pendulum

$$\boxed{\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + \{\Omega + h\,\cos(\mathrm{t})\}\,Y = 0}$$

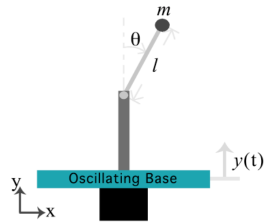
Faraday 1831



Kapitza 1951

$\Omega > 0$: Oscillator whose frequency $\sqrt{\Omega}$ is modulated   Prametric instability (Faraday 1831)

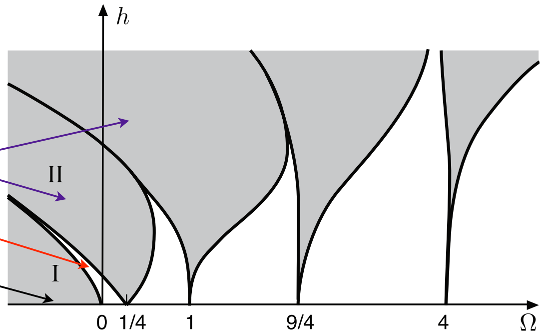
$\Omega < 0$: Re-stabilization of an unstable position of a pendulum by oscillations (Kapitza 1951)



Parametric instability

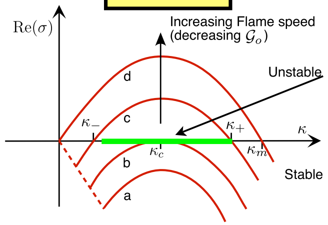
Re-stabilization tongue

Unstable situation



Stability limits of the solutions to Mathieus's equation
White regions: stable. Grey regions unstable

$u_a = 0$



unstable wavelengths

Flame propagating downward

$$\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}\tau'^2} + 2\mathsf{B}\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}\tau'} + \left[-\mathsf{D} + \varpi^2\mathsf{C}\cos(\varpi\tau')\right]\tilde{\alpha} = 0$$

$g'(t) = \omega u_a U_L \cos(\omega t)$

Markstein 1960

$u_{aI}^{*2} \approx 2\upsilon_b \dfrac{(\upsilon_b+1)}{(\upsilon_b-1)}\left(1-\dfrac{U_{Lc}}{U_L}\right), \qquad \dfrac{k_I^*}{k_m} \approx \dfrac{1}{2}\dfrac{U_{Lc}}{U_L}$

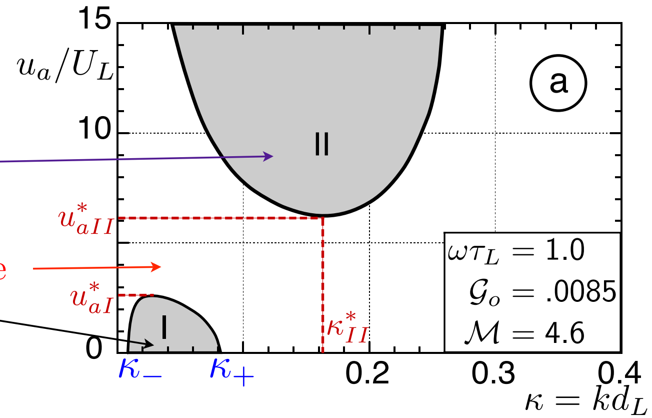
$u_{aII}^* = 2\upsilon_b/(\upsilon_b - 1)$

Bychkov 1999   Clavin 2015
ok with experiments Searby Rochewerger 1991

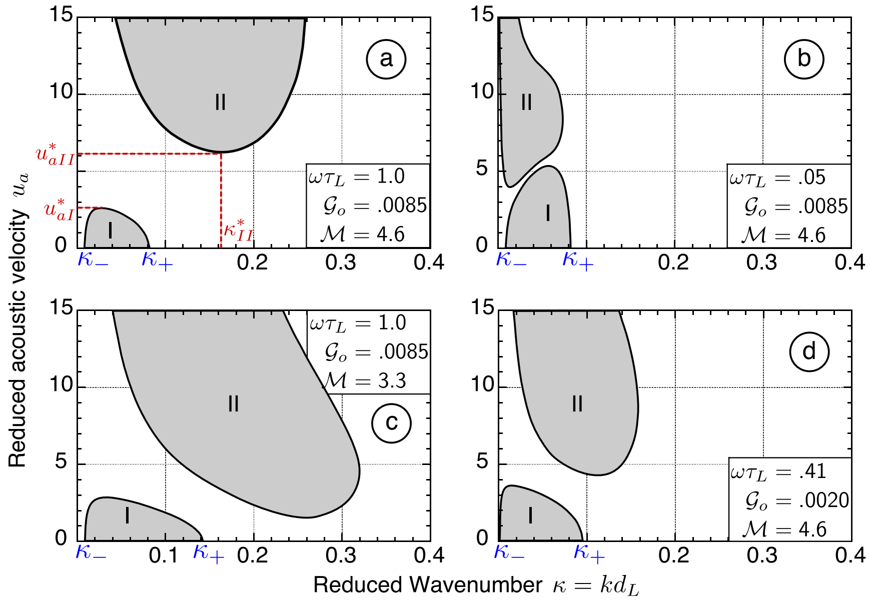
Parametric instability

$u_{aI}^* < u_a < u_{aII}^*$
Re-stabilization tongue

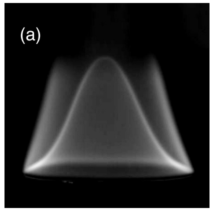
unstable wavelengths



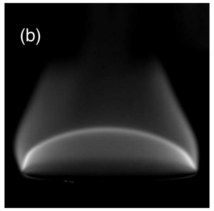
Sensitivity of the acoustic instability to the Markstein number an the acoustic frequency



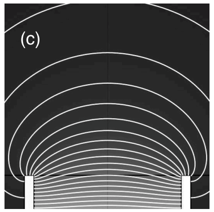
Flattening of Bunsen flames in an acoustic field (Hahnemann Ehret 1943, Durox et al. 1997, Baillot et al. 1999)



Rich Bunsen methane flame



+ intense axial acoustic field
140 Hz



acoustic equipotential surface