

*Tsinghua-Princeton-CI Summer School*  
*July 19-25, 2016*

**Lectures on**  
**Dynamics of Combustion Waves**  
**in Premixed Gases**

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**Lecture X**  
**Supersonic waves**

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## Lecture 10 : **Supersonic waves**

### 10-1. Background

*Model of hyperbolic equations for the formation of discontinuity*

*Riemann invariants*

*Rankine-Hugoniot conditions for shock waves*

*Mikhelson (Chapman-Jouguet) conditions for detonations*

### 10-2. Inner structure of a weak shock wave

*Formulation*

*Dimensional analysis*

*Analysis*

### 10-3. ZND structure of detonations

### 10-4. Selection mechanism of the CJ wave

# X-I) Background



Riemann 1860

shock wave  $\approx$  discontinuity in the solution of the Euler equations

## Model of hyperbolic equations for the formation of discontinuities

$$a(u) \quad \boxed{\partial u / \partial t + a(u) \partial u / \partial x = 0}$$

$$t = 0 : \quad u = u_o(x) \qquad u = u(x, t) ?$$

Simple case: linear equation

$$a = a_o = \text{cst.} :$$

$$\boxed{u = u_o(x - a_o t)}$$

propagation at constant velocity **without deformation**

Nonlinear equation  $a(u) \quad da/du \neq 0$

### *Method of characteristics*

The solution is conserved along any trajectory  $dx/dt = a(u)$  in the phase plan  $(x, t)$ .

$$u = u(x(t), t) \qquad du/dt = 0$$

$$u(x, t) = \text{cst.} \Rightarrow a(x, t) = \text{cst.}$$

$u(x, t)$  is **constant** along the **straight lines**  $\boxed{x = a_o t + x_o}$ :  $u = u_o$

$$\partial u / \partial t + a(u) \partial u / \partial x = 0$$

$$t = 0 : u = u_o(x)$$

$$u = u(x, t) ?$$



Riemann 1860

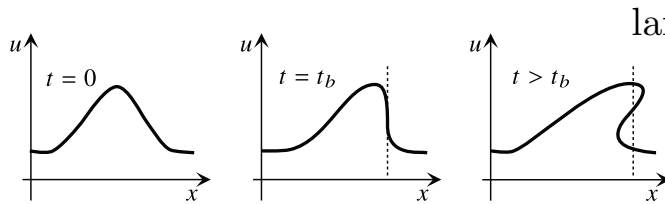
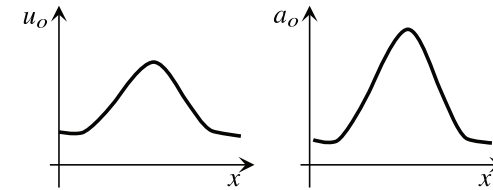
### Method of characteristics

The solution is conserved along any trajectory  $dx/dt = a(u)$  in the phase plan  $(x, t)$ .

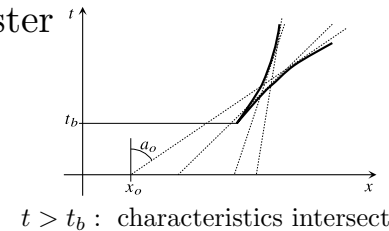
$u(x, t)$  is constant along the straight lines  $x = a_o t + x_o$ :  $u = u_o$

Speed increases with increasing  $u$ ,  $da/du > 0$

⇒ formation of singularities after a finite time



larger values run faster



$t > t_b$ : multivalued solution. Wave breaking

$$u(x, t) = u_o(x_o(x, t)), \quad x(x_o, t) = a_o(x_o)t + x_o, \quad \partial x / \partial x_o = 1 + t(da_o/dx_o) \quad \partial x_o / \partial x = [1 + t(da_o/dx_o)]^{-1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial x_o}{\partial x} \frac{du_o}{dx_o} \quad \frac{\partial u}{\partial x} = \frac{du_o/dx_o}{[1 + t(da_o/dx_o)]} \quad \text{diverges at time } t = \frac{1}{-da_o/dx_o} \quad \text{where } da_o/dx_o < 0$$

$t_b \equiv$  time of wave breaking (shortest time for the divergence of  $\partial u / \partial x$ )

$$t_b = \frac{1}{\max |da_o/dx_o|}$$

### Discontinuous solutions

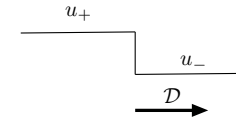
$$\partial u / \partial t + a(u) \partial u / \partial x = 0 \quad \text{conservative form} \quad \partial u / \partial t + \partial j / \partial x = 0 \quad j(u) \quad dj / du = a(u)$$

Are step functions  $u_+ \neq u_-$  propagating at constant velocity  $\mathcal{D}$  solutions ?

$$\partial / \partial t = -\mathcal{D} d / d\xi \quad \partial / \partial x = d / d\xi$$

$$u(\xi) \quad \xi = x - \mathcal{D}t$$

$$-\mathcal{D} \frac{du}{d\xi} + \frac{dj}{d\xi} = 0 \quad j - \mathcal{D}u \quad \text{is a conserved scalar}$$

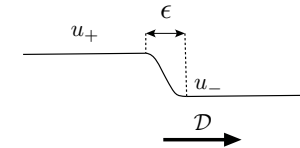


$$\mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-}$$

Infinite numbers of solutions !! Ill posed problem

$$f(u) \times \left( -\mathcal{D} \frac{du}{d\xi} + \frac{dj}{d\xi} \right) = 0 \quad G - \mathcal{D}F = \text{cst.} \quad \mathcal{D} = \frac{G(u_+) - G(u_-)}{F(u_+) - F(u_-)} \quad \text{where} \quad \frac{dF}{du} \equiv f(u) \quad \frac{dG}{du} \equiv f(u)a(u)$$

Adding a small dissipative term makes the problem well posed



$$\partial u / \partial t + a(u) \partial u / \partial x = \epsilon \partial^2 u / \partial x^2, \quad \epsilon > 0$$

$$\frac{d}{d\xi} \left[ -u\mathcal{D} + j(u) - \epsilon \frac{du}{d\xi} \right] = 0 \quad \epsilon \frac{du}{d\xi} = j(u) - u\mathcal{D} + \text{cst} \quad \frac{\xi}{\epsilon} = \int_0^u \frac{du}{j(u) - u\mathcal{D} + \text{cst}}$$

$\xi = \pm\infty : du/d\xi = 0 \Rightarrow$  2 expressions of the cst that should be equal  $\Rightarrow$  a single value of  $\mathcal{D}$

$$\mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-}$$

independent of  $\epsilon$  !

$$j(u_+) - u_+\mathcal{D} + \text{cst} = 0$$

$$j(u_-) - u_-\mathcal{D} + \text{cst} = 0$$

$u(\xi)$  continuous function

$\lim_{\epsilon \rightarrow 0} u = \text{step function}$

$a(u) = u$  : Burgers equation. Analytical solution to the initial value problem

## Riemann invariants (1860)

Euler equation + constant entropy + ideal gas:  $a^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho} \quad \frac{p}{\rho^\gamma} = \text{cst} \quad 2 \frac{da}{a} = \frac{dp}{p} - \frac{d\rho}{\rho} \Rightarrow 2 \frac{da}{a} = (\gamma - 1) \frac{d\rho}{\rho}$

$$\lambda \times \frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + \rho \frac{\partial}{\partial x} u = 0 \quad \frac{a^2}{\rho} \frac{\partial}{\partial x} \rho + \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u = 0$$

$$\lambda \frac{\partial}{\partial t} \rho + \left( \lambda u + \frac{a^2}{\rho} \right) \frac{\partial}{\partial x} \rho + \frac{\partial}{\partial t} u + (\lambda \rho + u) \frac{\partial}{\partial x} u = 0$$

choose  $\lambda$  such that  $\lambda u + a^2/\rho = \lambda(\lambda \rho + u), \quad \lambda = \pm a/\rho \quad \lambda \rho = \pm a$

$$\lambda \left[ \frac{\partial}{\partial t} + (\lambda \rho + u) \frac{\partial}{\partial x} \right] \rho + \left[ \frac{\partial}{\partial t} + (\lambda \rho + u) \frac{\partial}{\partial x} \right] u = 0$$

$$\lambda \rho + u = u \pm a$$

2 characteristics :

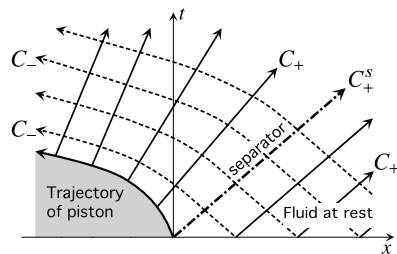
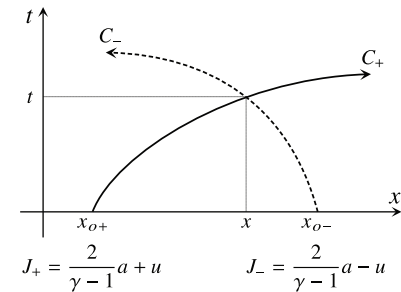
$$C_+ : \frac{dx}{dt} = u + a, \quad C_- : \frac{dx}{dt} = u - a$$

invariants :

$$C_+ : \frac{a}{\rho} d\rho + du = 0 \quad C_- : \frac{a}{\rho} d\rho - du = 0$$

$$\lambda d\rho + du = 0 \quad \pm \frac{a}{\rho} d\rho + du = 0$$

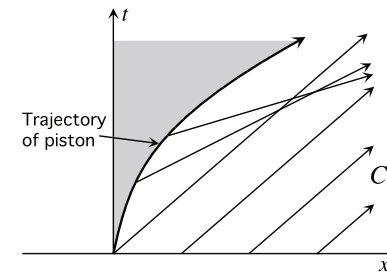
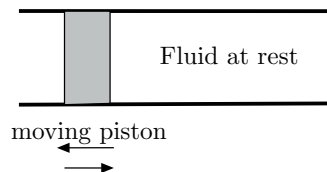
$$J_+ \equiv \frac{2}{\gamma - 1} a + u = \text{cst sur } C_+ \quad J_- \equiv \frac{2}{\gamma - 1} a - u = \text{cst sur } C_-$$



Rarefaction wave

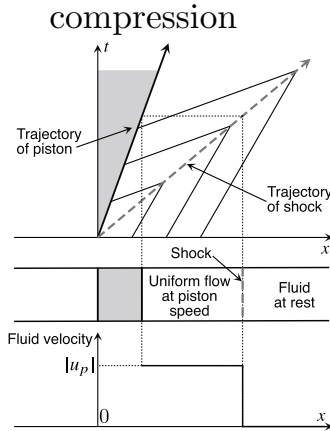
### Simple waves

$C_+$  are straight lines



Compression wave

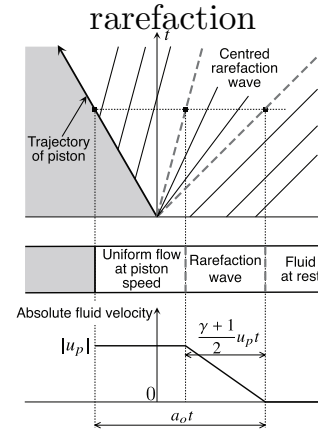
formation of a singularity: shock wave



**Centred waves**

$t < 0$  : piston velocity = 0  
 $t > 0$  : piston velocity = cst  $\neq 0$

Sel-similar solutions  $x/t$



no discontinuity of u

discontinuity of  $du/dx$  propagating at the local sound speed

fully unsteady process: thickness  $\nearrow u_p t$

shock wave at constant velocity  $>$  piston velocity



Rankine 1870



Hugoniot 1880

**Rankine-Hugoniot conditions for shock waves**  
 (1870 – 1880)

Eqs for the conservation of mass, momentum and energy

Initial state $\rho_u \quad p_u$ $\mathcal{D}$ $\longrightarrow$	Shocked gas $\rho_N \quad p_N$ $u_N$ $\longrightarrow$	$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$	$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} (p + \rho u^2 - \mu \frac{\partial u}{\partial x})$	$\frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} [\rho u(h + u^2/2) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x}]$
		$m \equiv \rho_u \mathcal{D} = \rho_N u_N$	$p_u + \frac{m^2}{\rho_u} = p_N + \frac{m^2}{\rho_N}$	$h_N - h_u + (u_N^2 - \mathcal{D}^2)/2 = 0$

written in the moving frame of the shock at velocity  $\mathcal{D}$   
 steady problem

$$p_u - p_N = m^2 \left[ \frac{1}{\rho_N} - \frac{1}{\rho_u} \right] \quad h_u - h_N = \frac{m^2}{2} \left[ \frac{1}{\rho_N^2} - \frac{1}{\rho_u^2} \right]$$

$$h(\rho_u, p_u) - h(\rho_N, p_N) + \frac{1}{2} \left( \frac{1}{\rho_u} + \frac{1}{\rho_N} \right) (p_N - p_u) = 0$$

<p>Hugoniot curve <math>(p - 1/\rho)</math></p> $h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left( \frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$	<p>Michelson-Rayleigh line</p> $p - p_u = -m^2 \left( \frac{1}{\rho} - \frac{1}{\rho_u} \right)$
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**Ideal (polytropic) gas**  $\gamma \equiv c_p/c_v$

$$p = (c_p - c_v)\rho T, \quad h = c_p T = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

$$h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left( \frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$$

$$p - p_u = -m^2 \left( \frac{1}{\rho} - \frac{1}{\rho_u} \right)$$

$$\mathcal{P} \equiv \frac{(\gamma + 1)}{2\gamma} \left( \frac{p}{p_u} - 1 \right), \quad \mathcal{V} \equiv \frac{(\gamma + 1)}{2} \left( \frac{\rho_u}{\rho} - 1 \right)$$

$$(\mathcal{P} + 1)(\mathcal{V} + 1) = 1 \quad \mathcal{P} = -M_u^2 \mathcal{V}$$

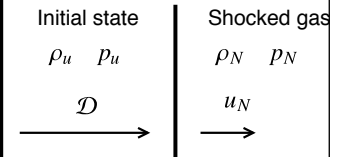
Hugoniot curve      Michelson-Rayleigh line

quadratic equation for  $\mathcal{V}$ , 2 solutions:  $\mathcal{V} = 0, \mathcal{V} = \mathcal{V}_N$

**Shocked gas (Neumann state) vs  $M_u$**

$$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma - 1)M_u^2 + 2}{(\gamma + 1)M_u^2} \quad \frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma - 1)}{(\gamma + 1)} \quad \frac{T_N}{T_u} = \frac{[2\gamma M_u^2 - (\gamma - 1)][(\gamma - 1)M_u^2 + 2]}{(\gamma + 1)^2 M_u^2}$$

$$M_N^2 = \frac{(\gamma - 1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma - 1)} \Leftrightarrow 2\gamma M_u^2 M_N^2 - (\gamma - 1)(M_u^2 + M_N^2) - 2 = 0$$



**General comments**

The Hugoniot curve is tangent to the isentropic

the entropy change along the Hugoniot curve is of third order  $\Rightarrow M_u \equiv \mathcal{D}/a_u > 1$

$$\delta s = s - s_u \quad \delta p = p - p_u$$

$$\delta s \nearrow \delta p \nearrow$$

$$\delta s = \frac{1}{12} \frac{1}{T_u} \left( \frac{\partial^2(1/\rho)}{\partial p^2} \right)_s (\delta p)^3$$

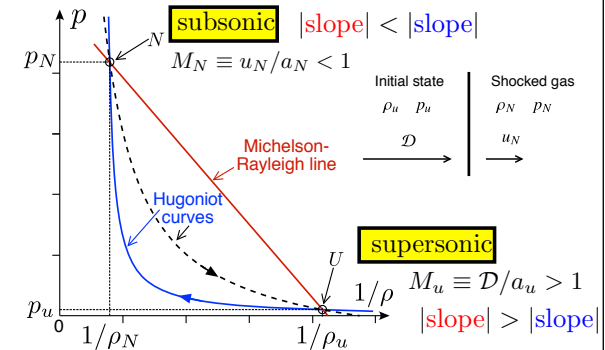
The Hugoniot relation is not an iso-function of state

$$h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left( \frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0 \quad \text{cannot be written in the form} \quad \mathcal{H}(1/\rho, p) = \mathcal{H}(1/\rho_u, p_u)$$

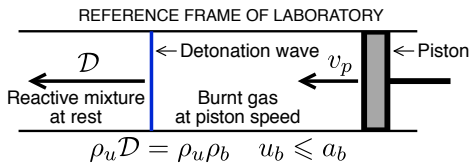
$$h(p, \rho) - h(p_N, \rho_N) - \frac{1}{2} \left( \frac{1}{\rho_N} + \frac{1}{\rho} \right) (p - p_N) = 0 \neq h(p, \rho) - h(p_u, \rho_u) - \frac{1}{2} \left( \frac{1}{\rho_u} + \frac{1}{\rho} \right) (p - p_u) = 0$$

Rarefaction shock does not exist. The entropy of the fluid increases through the shock (Irreversibility)

can be proved for weak shock by the entropy balance  $\rho u \frac{ds}{dx} = \frac{d}{dx} \left( \frac{\lambda}{T} \frac{dT}{dx} \right) + \dot{\omega}_s \quad \dot{\omega}_s > 0$   
 or by the H-theorem using the Boltzmann equation

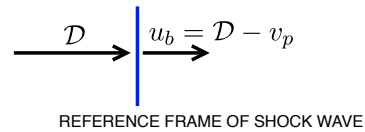


# Mikhelson condition for the CJ detonation (1893)



$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}, \quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} \left( p + \rho u^2 - \mu \frac{\partial u}{\partial x} \right), \quad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} \left[ \rho u \left( c_p T + \frac{u^2}{2} - q_m \psi \right) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x} \right]$$

$$c_p(T_b - T_u) + (u_b^2 - D^2)/2 = q_m$$



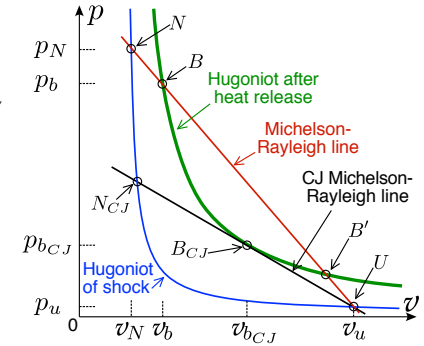
$$\frac{\gamma}{\gamma - 1} \left( \frac{p_b}{\rho_b} - \frac{p_u}{\rho_u} \right) - \frac{1}{2} (p_b - p_u) \left( \frac{1}{\rho_u} + \frac{1}{\rho_b} \right) = q_m$$

$$\mathcal{P} \equiv \frac{(\gamma + 1)}{2\gamma} \left( \frac{p}{p_u} - 1 \right), \quad \mathcal{V} \equiv \frac{(\gamma + 1)}{2} \left( \frac{\rho_u}{\rho} - 1 \right)$$

$$\mathcal{Q} \equiv \frac{\gamma + 1}{2} \frac{q_m}{c_p T_u}$$

$$(\mathcal{P} + 1)(\mathcal{V} + 1) = 1 + \mathcal{Q}$$

$$\mathcal{P} = -M_u^2 \mathcal{V}$$



quadratic equation for  $\mathcal{V}$   $M_u^2 \mathcal{V}^2 + (M_u^2 - 1)\mathcal{V} + \mathcal{Q} = 0$

supersonic combustion wave  $M_u \equiv D/a_u > 1$

**Lower bound of propagation velocity  $D = D_{CJ}$**   
(called Chapmann-Jouguet 1899 – 1904)

$$(M_u^2 - 1)^2 \geq 4\mathcal{Q}M_u^2 \quad M_u \geq M_{u_{CJ}} \equiv \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q} + 1}, \quad \text{Mikhelson (1893)}$$

In the CJ wave the velocity of the burned gas is sonic in the frame of the wave  $u_{bCJ} = a_{bCJ}$  (self-sustained wave)  
(Rayleigh line is tangent)

In the overdriven detonations  $D > D_{CJ}$  the velocity of the burned gas is subsonic in the frame of the wave  $u_b < a_b$   
(piston-supported detonation)

## Vielle conjecture (1900)

detonation = inert shock wave followed by a exothermal reaction zone

$$U \rightarrow N \rightarrow B$$

(large Arrhenius factor)



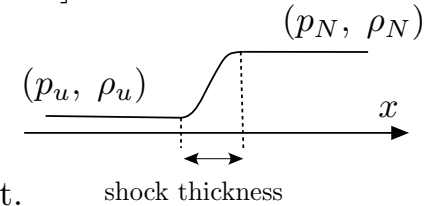
Paul Vieille (1900)

# X-2) Inner structure of a weak shock

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x} \left( p + \rho u^2 - \mu \frac{\partial u}{\partial x} \right) \quad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x} \left[ \rho u \left( h + \frac{u^2}{2} \right) - \lambda \frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x} \right]$$

## Formulation

( reference frame attached to the shock wave )



$$\rho u = m, \quad p + \rho u^2 - \mu \frac{du}{dx} = \text{cst.} \quad m \left( h + \frac{u^2}{2} \right) - \lambda \frac{dT}{dx} - \mu u \frac{du}{dx} = \text{cst.}$$

$$x \rightarrow -\infty : \quad p = p_u, \quad \rho = \rho_u, \quad u = \mathcal{D} \quad x \rightarrow \infty : \quad dp/dx = 0, \quad d\rho/dx = 0, \quad du/dx = 0$$

$$\frac{p}{\rho} = (\gamma - 1)c_v T \quad h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

Two coupled equations for  $p$  and  $v \equiv 1/\rho, m$  given ( $u = mv, \quad c_v T = pv/(\gamma - 1)$ )

$$(p - p_u) + m^2(v - v_u) = \mu m \frac{dv}{dx},$$

$$\frac{\gamma}{\gamma - 1}(pv - p_u v_u) - \frac{1}{2}(p - p_u)(v + v_u) = \frac{\gamma}{\gamma - 1} \frac{\lambda}{m c_p} \frac{d(pv)}{dx} + \frac{\mu m}{2}(v - v_u) \frac{dv}{dx}$$

$$x \rightarrow \infty : \quad dp/dx = 0, \quad dv/dx = 0$$

$$\frac{1}{2}(u^2 - u_u^2) = \frac{m^2}{2}(v + v_u)(v - v_u) = \frac{1}{2}(v + v_u) \left[ -(p - p_u) + \mu m \frac{dv}{dx} \right]$$

## Dimensional analysis

speed of sound  
 $u/a = O(1),$

mean free path  
 $\mu/\rho = \text{viscous diffusion coefficient} \approx \ell/a$   
 kinetic theory of gases

$$\rho u^2 \quad \times \quad \mu \frac{du}{dx}$$

$\Rightarrow$  thickness of shock waves  $\approx$  mean free path

macroscopic equations not valid ?

ok for weak shock !

$M_u \equiv \mathcal{D}/a_u > 1$     **Analysis for**  $\epsilon \equiv M_u - 1 \ll 1$     (weak shock)

$v \equiv 1/\rho$      $\nu \equiv (v - v_u)/v_u = O(\epsilon)$      $\pi \equiv (p - p_u)/p_u = O(\epsilon)$

mean free path

*Non dimensional equations*

$\xi \equiv x/\ell$      $\ell \equiv D_{Tu}/a_u$      $a_u = \sqrt{\gamma p_u/\rho_u}$      $Pr \equiv \mu/(\rho_u D_{Tu})$      $M_u^2 = 1 + 2\epsilon + ..$

$\frac{\gamma}{\gamma-1}(pv - p_u v_u) - \frac{1}{2}(p - p_u)(v + v_u) = \frac{\gamma}{\gamma-1} \frac{\lambda}{m c_p} \frac{d(pv)}{dx} + \frac{\mu m}{2} \frac{d(v - v_u)}{dx} + O(\epsilon^3)$

$\frac{m^2 v_u}{\gamma p_u} = M_u^2 \approx 1 + 2\epsilon$

anticipating  $\left\{ \begin{array}{l} d\nu/d\xi = O(\epsilon^2) \\ d\pi/d\xi = O(\epsilon^2) \end{array} \right\}$      $\frac{1}{\gamma}\pi + (1 + 2\epsilon)\nu = Pr \frac{d\nu}{d\xi} + O(\epsilon^3)$ ,    valid up to order  $\epsilon^2$

$\left( \frac{\gamma+1}{2\gamma} \right) \pi\nu + \frac{1}{\gamma}\pi + \nu = \frac{d\pi}{d\xi} + \frac{d\nu}{d\xi} + O(\epsilon^3)$ ,

$\frac{1}{\gamma}\pi + \nu = Pr \frac{d\nu}{d\xi} - 2\epsilon\nu + O(\epsilon^3)$ ,     $\Rightarrow$      $\left( \frac{\gamma+1}{2\gamma} \right) \pi\nu + Pr \frac{d\nu}{d\xi} - 2\epsilon\nu = \frac{d\pi}{d\xi} + \frac{d\nu}{d\xi} + O(\epsilon^3)$ ,

$\pi = -\gamma\nu + O(\epsilon^2)$ ,     $\Rightarrow$      $\boxed{[(\gamma - 1) + Pr] \frac{d\nu}{d\xi} = \left( \frac{\gamma+1}{2} \nu + 2\epsilon \right) \nu}$

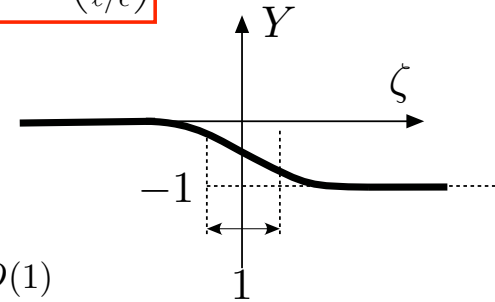
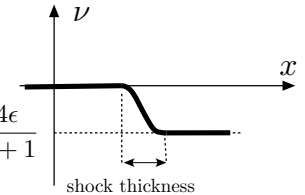
the Rankine-Hugoniot jumps at order  $\epsilon$  are recovered when dissipative terms are neglected     $\nu_N = -4\epsilon \frac{1}{\gamma+1}$      $\pi_N = -4\epsilon \frac{\gamma}{\gamma+1}$

$\frac{2}{\gamma+1} [(\gamma - 1) + Pr] \frac{d\nu}{d\xi} = \nu(\nu - \nu_N) \leq 0$

$\xi = -\infty$  : initial state,  $\nu = 0$ ,     $\xi = +\infty$  : shocked gas,  $\nu = \nu_N = -4\epsilon/(\gamma + 1)$

$Y \equiv \frac{(\gamma+1)}{4\epsilon} \nu \in [0, -1]$      $\zeta \equiv \frac{2}{[(\gamma+1) + Pr]} \epsilon \xi = \frac{2}{[(\gamma+1) + Pr]} \frac{x}{(\ell/\epsilon)}$      $\boxed{\zeta = O\left(\frac{x}{\ell/\epsilon}\right)}$

$\boxed{\frac{dY}{d\zeta} = Y(Y + 1) < 0}$   
 $\zeta = -\infty : Y = 0, \quad \zeta = +\infty : Y = -1$



$\zeta = \int \frac{dY}{Y(Y+1)}$      $Y(\zeta) = -\frac{e^\zeta}{e^\zeta + 1}$

shock thickness = mean free path /  $(M_u - 1)$

microscopic length if  $M_u - 1 = O(1)$   
 macroscopic length if  $(M_u - 1) \ll 1$

# X-3) ZND structure of detonations

Zeldovich (1940) Neumann (1942) Döring (1944)

## Orders of magnitude

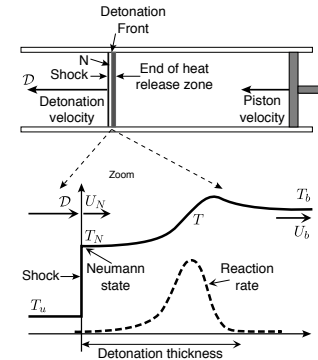
$$\frac{E}{k_B T_N} \gg 1 \Rightarrow \frac{1}{\tau_r(T_N)} \approx \frac{e^{-E/k_B T_N}}{\tau_{coll}} \ll \frac{1}{\tau_{coll}} \quad \frac{u_N}{a_N} = O(1), \quad d_N \equiv u_N \tau_r(T_N) \gg a_N \tau_{coll} \approx \ell$$

thickness of the reaction zone  $\gg$  thickness of the lead (inert) shock

structure of the detonation: **inert shock followed by a much larger reaction zone** conjectured by Vieille (1900)

$$\frac{D_T}{d_N^2} \approx \frac{a_N^2 \tau_{coll}}{d_N^2} \approx \frac{\tau_{coll}}{(\tau_r(T_N))^2} \approx \frac{e^{-E/k_B T_N}}{\tau_r(T_N)} \ll \frac{1}{\tau_r(T_N)}$$

diffusion rate  $\ll$  reaction rate  $\Rightarrow$  **diffusion terms are negligible**



## Formulation

Reference frame of the lead shock ( $x = 0$ )

$$\frac{d(\rho u)}{dx} = 0 \quad \frac{dp}{dx} + \rho u \frac{du}{dx} = 0$$

$$\frac{\gamma}{\gamma - 1} \frac{d}{dx} \left( \frac{p}{\rho} \right) + u \frac{du}{dx} - q_m \frac{d\psi}{dx} = 0 \quad \rho u \frac{d\psi}{dx} = \rho \frac{\dot{w}(T, \psi)}{\tau_r(T_N)}$$

$$\psi \in [0, 1] \quad \begin{aligned} \dot{w}(T, \psi = 1) &= 1 \\ \dot{w}(T, \psi = 0) &= 0. \end{aligned}$$

$$x = 0 : \quad u = u_N, \quad \rho = \rho_N, \quad p = p_N, \quad \psi = 1, \quad \dot{w} = 1$$

$$x \rightarrow \infty : \quad u = u_b, \quad \rho = \rho_b, \quad p = p_b, \quad \psi = 0, \quad \dot{w} = 0$$

detonation thickness  $d_N = u_N \tau_r(T_N)$

$a^2 = \gamma \frac{p}{\rho}$  Elimination of  $p$  and  $\rho$

$$\frac{d}{dx} \left( \frac{p}{\rho} \right) = p \frac{d}{dx} \left( \frac{1}{\rho} \right) + \frac{1}{\rho} \frac{dp}{dx} = \frac{p}{\rho u} \frac{du}{dx} + \frac{1}{\rho} \frac{dp}{dx} = \frac{a^2}{\gamma u} \frac{du}{dx} - u \frac{du}{dx} \Rightarrow \frac{\gamma}{\gamma - 1} \frac{d}{dx} \left( \frac{p}{\rho} \right) + u \frac{du}{dx} = \frac{1}{(\gamma - 1)u} (a^2 - u^2) \frac{du}{dx}$$

$$(a^2 - u^2) \frac{du}{dx} = (\gamma - 1) q_m u \frac{d\psi}{dx} \Rightarrow \frac{du^2}{d\psi} = 2(\gamma - 1) q_m \frac{u^2}{(a^2 - u^2)}$$

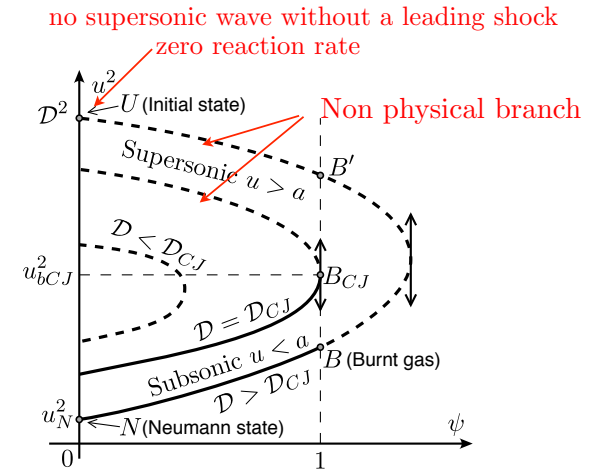
Phase portrait in the plan  $\psi - u^2$

$$\frac{du^2}{d\psi} = 2(\gamma - 1)q_m \frac{u^2}{(a^2 - u^2)}$$

$$\frac{1}{\gamma - 1}a^2 + \frac{1}{2}u^2 - q_m\psi = \frac{1}{\gamma - 1}a_u^2 + \frac{1}{2}D^2$$

Initial state  $\psi = 0$ :  $u^2 = D^2$ ,  $a^2 = a_u^2$

Neumann state  $\psi = 0$ :  $u^2 = u_N^2$ ,  $a^2 = a_N^2$



## X-4) Selection mechanism of the CJ wave

Rarefaction wave in the burnt gas when the piston is suddenly stopped

