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Lectures on Dynamics of Combustion Waves in Premixed Gases

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Lecture XII
Galloping detonations

Lecture 12 : Galloping detonations

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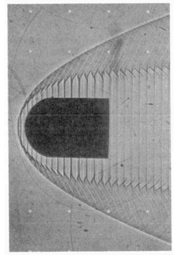
Slow time scale
Asymptotic model for CJ or near CJ regimes
Results for simplified chemical kinetics

XII-1) Physical mechanisms

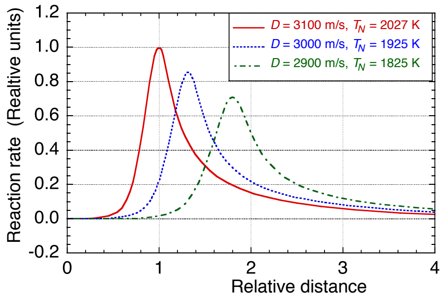


Lehr 1972

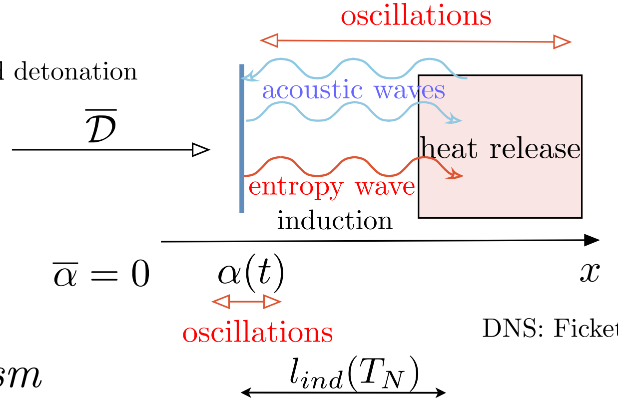
Detonation = inner shock followed by an exothermal reaction zone

Inner structure = uniform induction zone + zone of heat release

Galloping detonation = oscillatory instability: oscillation of the velocity of the lead shock



reference frame of the unperturbed detonation



DNS: Ficket Woods 1966

Instability mechanism

$\delta\mathcal{D} = -\dot{\alpha}_t \neq 0 \Rightarrow \delta T_N \Rightarrow \delta l_{ind}$ motion of the heat release zone produces a piston like effect

feedback loop

the unstable character depends on the phase shift

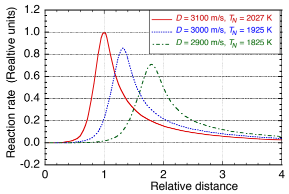
Two different coupling mechanisms: $\begin{cases} \text{acoustic waves} \\ \text{entropy wave} \end{cases}$

Two limiting cases

Strongly overdriven regimes for $(\gamma - 1) \ll 1$: quasi-isobaric flow, the delay by the acoustic waves is negligible

CJ regime for $q_m/c_pT_u \ll 1$ and $(\gamma - 1) \ll 1$: transonic flow, the entropy wave is negligible
(not realistic but useful for pointing out the compressible effects)

XII-2) General formulation

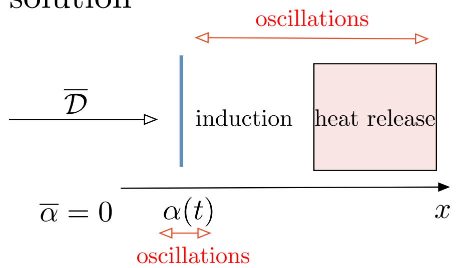


Galloping detonation = pulsating instability of the 1-D solution

velocity of the shock oscillates

reaction rate oscillates

$$\boxed{\mathcal{D}(t) = \overline{\mathcal{D}} - \dot{\alpha}_t,}$$



Constitutive equations

Reactive Euler equations

$$\mathrm{D}/\mathrm{Dt} \equiv \partial/\partial t + \mathbf{u}.\nabla \qquad \rho c_p \mathrm{D}T/\mathrm{Dt} = \mathrm{D}p/\mathrm{Dt} + \rho q_m \dot{\mathrm{w}}$$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{Dt}} = -\nabla.\mathbf{u}, \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{Dt}} = -\nabla p, \qquad p = (c_p - c_v)\rho T \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N}, \qquad \frac{\mathrm{D}\psi}{\mathrm{Dt}} = \frac{\dot{\mathrm{w}}(\psi, T)}{\bar{t}_N} \qquad \bar{t}_N \equiv \tau_r(\overline{T}_N)$$

Reduced mass weighted distance from the shock (useful for unsteady 1-D problems)

$$\mathrm{x} \equiv \frac{1}{\rho_u \overline{\mathcal{D}}\bar{t}_N}\int_{\alpha(t)}^{x} \rho(x', t)\mathrm{d}x', \qquad \mathrm{t} \equiv \frac{t}{\bar{t}_N}, \qquad \bar{t}_N \equiv \tau_r(\overline{T}_N)$$

$x = \alpha(t)$ instantaneous shock position

reaction time at the Neumann state of the unperturbed solution

$\dot{\alpha}_t \equiv \mathrm{d}\alpha/\mathrm{dt}$

$$\frac{\partial}{\partial x} = \frac{\rho}{\rho_u}\frac{1}{\overline{\mathcal{D}}\bar{t}_N}\frac{\partial}{\partial \mathrm{x}} \qquad \bar{t}_N\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial \mathrm{t}} + \mathrm{m}(\mathrm{t})\frac{\partial}{\partial \mathrm{x}}, \qquad \text{where} \qquad \mathrm{m}(\mathrm{t}) \equiv \left[\frac{\rho(x,t)[u(x,t) - \dot{\alpha}_t]}{\rho_u \overline{\mathcal{D}}}\right]_{x=\alpha(t)} = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$\begin{cases} u|_{\mathrm{x}=0}(t) - \dot{\alpha}_t = u_N(t) \\ \rho_N(t) u_N(t) = \rho_u(\overline{\mathcal{D}} - \dot{\alpha}_t) \end{cases}$$

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial \mathrm{t}} + \mathrm{m}(\mathrm{t})\frac{\partial}{\partial \mathrm{x}} \qquad \mathrm{m}(\mathrm{t}) = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\rho_u}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{\mathcal{D}}}\right) \quad \Leftrightarrow \quad \frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\overline{\rho}_N}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{u}_N}\right),$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{u}{\overline{\mathcal{D}}}\right) = -\frac{\partial}{\partial \mathrm{x}}\left(\frac{p}{\rho_u \overline{\mathcal{D}}^2}\right), \qquad p = (c_p - c_v)\rho T,$$

$$\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_p T}\dot{\mathrm{w}}, \qquad \frac{\mathrm{D}\psi}{\mathrm{Dt}} = \dot{\mathrm{w}},$$

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial \mathrm{t}} + \mathrm{m(t)}\frac{\partial}{\partial \mathrm{x}} \qquad \mathrm{m(t)} = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\rho_u}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{\mathcal{D}}}\right) \quad \Leftrightarrow \quad \frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\overline{\rho}_N}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{u}_N}\right),$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{u}{\overline{\mathcal{D}}}\right) = -\frac{\partial}{\partial \mathrm{x}}\left(\frac{p}{\rho_u \overline{\mathcal{D}}^2}\right), \qquad p = (c_p - c_v)\rho T,$$

$$\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_p T}\dot{\mathrm{w}}, \qquad \frac{\mathrm{D}\psi}{\mathrm{Dt}} = \dot{\mathrm{w}},$$

$m(t)$ unknow

$\dot{\alpha}_t/\overline{\mathcal{D}} = 1 - m(t)$

Boundary conditions

Neumann state $\mathrm{x} = 0: \quad \rho = \rho_N(\mathrm{t}), \quad p = p_N(\mathrm{t}), \quad T = T_N(\mathrm{t}) \quad \rho_N(\mathrm{t})(u - \dot{\alpha}_t) = \rho_u \overline{\mathcal{D}}\,\mathrm{m(t)} \quad \psi = 0$

expressed in terms of m(t) by the RH conditions

$$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma-1)M_u^2 + 2}{(\gamma+1)M_u^2}, \qquad \frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma-1)}{(\gamma+1)},$$

$$\frac{T_N}{T_u} = \frac{\left[2\gamma M_u^2 - (\gamma-1)\right]\left[(\gamma-1)M_u^2 + 2\right]}{(\gamma+1)^2 M_u^2}, \qquad M_N^2 = \frac{(\gamma-1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma-1)}.$$

$$\boxed{M_u = \overline{M}_u \mathrm{m(t)}}$$

Burnt gas $\mathrm{x} \to \infty$: $\begin{cases} \text{overdriven regimes: } u = \overline{u}_b \\ \text{CJ regime: } p - \overline{p}_b = \overline{\rho}_b \overline{a}_b (u - \overline{u}_b) \end{cases}$ i.e. outgoing acoustic waves (radiation condition)

Analytical solutions are obtained in limiting cases

Strong shock in the Newtonian approximation

$$\overline{M}_u \gg 1, \quad (\gamma-1) \ll 1 \quad \Rightarrow \quad \overline{M}_N^2 \approx \frac{\gamma-1}{2} + \frac{1}{\overline{M}_u^2} \ll 1$$

Distinguished limit: $\overline{M}_u^2 \gg 1, \quad (\gamma-1)M_u^2 = O(1)$

$$\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma-1) = O(\epsilon^2)$$

small parameter

$$\overline{u}_N/\overline{\mathcal{D}} = \overline{\rho}_u/\overline{\rho}_N \approx \epsilon^2, \qquad \overline{p}_N/\overline{p}_u \approx \overline{M}_u^2 = O(1/\epsilon^2), \qquad \overline{a}_N/\overline{\mathcal{D}} \approx \epsilon, \qquad (\overline{a}_N/\overline{a}_u)^2 \approx [2 + (\gamma-1)\overline{M}_u^2]/2 = O(1)$$

$$\left.\begin{aligned} &\overline{\rho u} - \overline{\rho}_N \overline{u}_N = 0 \\ &\frac{\mathrm{d}\overline{p}}{\mathrm{d}x} + \overline{\rho u}\frac{\mathrm{d}\overline{u}}{\mathrm{d}x} = 0 \end{aligned}\right\} \quad \text{and} \quad a_N^2/\gamma = p_N/\rho_N \quad \Rightarrow \quad \boxed{(\overline{p}/\overline{p}_N - 1) = -\epsilon^2\,(\overline{u}/\overline{u}_N - 1)\,,}$$

Quasi-isobaric approximation of the shocked gas

XII-3) Strongly overdriven detonations in the limit $(\gamma-1)\ll 1$

(Clavin He 1996)

Distinguished limit

$$\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma-1) = O(\epsilon^2)$$

$$\boxed{q_N \equiv q_m/c_p\overline{T}_N = O(1)} \quad \Leftrightarrow \quad \text{strongly overdriven regime}$$

$M_{u_{CJ}} = \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q}+1}$ where $\mathcal{Q} \equiv \dfrac{\gamma+1}{2}\dfrac{q_m}{c_pT_u}$

$\overline{T}_N/T_u = O(1) \quad \Rightarrow M_{u_{CJ}} = O(1) \Rightarrow M_u \gg M_{u_{CJ}}$

$$\left.\begin{array}{r}\dfrac{1}{T}\dfrac{\mathrm{D}T}{\mathrm{Dt}} - \dfrac{(\gamma-1)}{\gamma}\dfrac{1}{p}\dfrac{\mathrm{D}p}{\mathrm{Dt}} = \dfrac{q_m}{c_pT}\dot{\mathrm{w}}, \\ \delta p/p = O(\epsilon^2)\end{array}\right\} \Rightarrow \boxed{\left\{\begin{array}{l}\dfrac{\partial T}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial T}{\partial \mathrm{x}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi,T), \\ \dfrac{\partial \psi}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial \psi}{\partial \mathrm{x}} = \dot{\mathrm{w}}(\psi,T)\end{array}\right.}$$

$$\mathrm{x}=0: \quad \psi = 0, \quad T = T_N(\mathrm{t})$$

$T_N/T_u \approx [(\gamma-1)M_u^2+2]/2$ where $M_u^2 = \overline{M}_u^2(1-\dot{\alpha}_t/\overline{\mathcal{D}})^2$

The solution yields T and p in terms of $m(t) = 1-\dot{\alpha}_t/\mathcal{D}$

Integral differential equation for the dynamics (Clavin He 1996)

Quasi-isobaric approximation in the shocked gas

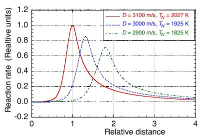
$$\left.\begin{array}{r}\dfrac{\mathrm{D}}{\mathrm{Dt}}\left(\dfrac{\overline{\rho}_N}{\rho}\right) = \dfrac{\partial}{\partial \mathrm{x}}\left(\dfrac{u}{\overline{u}_N}\right), \\ \delta p/p = O(\epsilon^2)\end{array}\right\} \Rightarrow \left(\dfrac{\partial}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial}{\partial \mathrm{x}}\right)\dfrac{T}{\overline{T}_N} = \dfrac{\partial}{\partial \mathrm{x}}\left(\dfrac{u}{\overline{u}_N}\right) \Rightarrow q_N\int_0^\infty \dot{\mathrm{w}}\,\mathrm{dx} = \dfrac{u_b}{\overline{u}_N} - \dfrac{u(\mathrm{x}=0)}{\overline{u}_N}$$

boundary condition in the burnt gas

$$\mathrm{x}=0: \quad \rho_N(t)(u-\dot{\alpha}_t) = \rho_u(\overline{\mathcal{D}}-\dot{\alpha}_t) \Rightarrow \dfrac{u(\mathrm{x}=0)}{\overline{u}_N} = \left(1-\cancel{\dfrac{\rho_u}{\rho_N(t)}}\right)\dfrac{\dot{\alpha}_t}{\overline{u}_N} + \dfrac{\rho_u\overline{\mathcal{D}}}{\rho_N(t)\overline{u}_N}$$

$\rho_u/\rho_N = O(\epsilon^2)$

Large T_N-sensitivity of the induction length $\beta_N \gg 1$ $\quad \beta_N \equiv \dfrac{T_N}{l_{ind}}\dfrac{\mathrm{d}l_{ind}}{\mathrm{d}T_N}$



$$\delta T_N/T_N = O(1/\beta_N) \Rightarrow \delta T(\mathrm{x,t})/T = O(1), \ \delta\psi(\mathrm{x,t})/\psi = O(1) \quad \delta\dot{\mathrm{w}} = O(1)$$

Attention is limited to $\delta M_u/\overline{M}_u = O(1/\beta_N)$

$$\left.\begin{array}{r}\delta\rho_N(t)/\overline{\rho}_N = O(1/\beta_N) \\ \rho_u\overline{\mathcal{D}}/\rho_N(t)\overline{u}_N = 1+O(1/\beta_N)\end{array}\right\} \Rightarrow \boxed{\left(\dfrac{\overline{u}_b}{\overline{u}_N}-1\right) - \dfrac{\dot{\alpha}_t}{\overline{u}_N} \approx q_N\int_0^\infty \dot{\mathrm{w}}\,\mathrm{dx}}$$

$\dot{\mathrm{w}}$ in terms of $\dot{\alpha}_t/\overline{u}_N$

$$m(t) = 1 - \dot{\alpha}_t/\overline{\mathcal{D}}, \quad \dot{\alpha}_t/\overline{\mathcal{D}} = O(1/\beta_N) \Rightarrow \begin{cases} \dfrac{\partial T}{\partial \mathrm{t}} + \dfrac{\partial T}{\partial \mathrm{x}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi, T), \\ \dfrac{\partial \psi}{\partial \mathrm{t}} + \dfrac{\partial \psi}{\partial \mathrm{x}} = \dot{\mathrm{w}}(\psi, T) \end{cases} \quad \mathrm{x} = 0: \ T = T_N(t), \ \psi = 0,$$

$$(T_N - \overline{T}_N)/\overline{T}_N = O(1/\beta_N) \ \Rightarrow \ \boxed{\Theta_N(\mathrm{t}) \equiv \beta_N(T_N(t) - \overline{T}_N)/\overline{T}_N = O(1),} \quad \dot{\mathrm{w}} - \dot{\overline{\mathrm{w}}} = O(1)$$

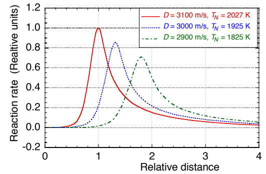
Steady state solutions

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{dx}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi, T), \\ \dfrac{\mathrm{d}\psi}{\mathrm{dx}} = \dot{\mathrm{w}}(\psi, T) \end{cases} \quad \mathrm{x} = 0: \ T = T_N, \ \psi = 0,$$

$T_N = \ \text{cst.} \neq \overline{T}_N$

$\Theta_N \equiv \beta_N(T_N - \overline{T}_N)/\overline{T}_N$

$$T = \mathcal{T}(\Theta_N, \mathrm{x}), \quad \psi = \mathcal{Y}(\Theta_N, \mathrm{x}), \quad \Omega(\Theta_N, \mathrm{x}) \equiv \dot{\mathrm{w}}(\mathcal{T}, \mathcal{Y})$$



Unsteady solution (retarded functions)

$$\boxed{T(\mathrm{x}, \mathrm{t}) = \mathcal{T}\left(\Theta_N(\mathrm{t} - \mathrm{x}), \mathrm{x}\right), \quad \psi(\mathrm{x}, \mathrm{t}) = \mathcal{Y}\left(\Theta_N(\mathrm{t} - \mathrm{x}), \mathrm{x}\right), \quad \dot{\mathrm{w}} = \Omega(\Theta_N(t - \mathrm{x}), \mathrm{x})}$$

$$\left(\frac{\overline{u}_b}{\overline{u}_N} - 1\right) - \frac{\dot{\alpha}_t}{\overline{u}_N} \approx q_N \int_0^\infty \dot{\mathrm{w}}\, \mathrm{dx} \quad \Rightarrow \quad \boxed{-\frac{\dot{\alpha}_t}{\overline{u}_N} = q_N \int_0^\infty \left[\Omega(\Theta_N(t - \mathrm{x}), \mathrm{x}) - \Omega(\overline{\Theta}_N, \mathrm{x})\right] \mathrm{dx}}$$

$\delta\Theta_N = O(1) \Rightarrow \delta\Omega = O(1)$

$\overline{u}_b/\overline{u}_N = q_N \int_0^\infty \Omega(\overline{\Theta}_N, \mathrm{x})\mathrm{dx}$

$$\left.\begin{aligned} \frac{T_N}{T_u} &= \frac{\left[2\gamma M_u^2 - (\gamma - 1)\right]\left[(\gamma - 1)M_u^2 + 2\right]}{(\gamma + 1)^2 M_u^2} \\ M_u^2 &= \overline{M}_u^2(1 - \dot{\alpha}_t/\overline{\mathcal{D}})^2 \end{aligned}\right\} \Rightarrow \ (T_N(\mathrm{t}) - \overline{T}_N)/\overline{T}_N \approx -(\gamma - 1)\dot{\alpha}_t/\overline{u}_N \ll 1, \quad \Rightarrow \quad \frac{\dot{\alpha}_t}{\overline{u}_N} = -\frac{\Theta_N(\mathrm{t})}{\beta_N(\gamma - 1)}$$

Distinguished limit $\boxed{(\gamma - 1)\beta_N = O(1)} \quad \Leftrightarrow \quad \boxed{\beta_N = O(1/\epsilon^2)}$

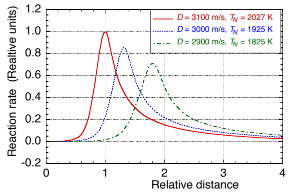
Nonlinear integral equation

$$\boxed{1 + b\Theta_N(\mathrm{t}) = \int_0^\infty \Omega(\Theta_N(\mathrm{t} - \mathrm{x}), \mathrm{x})\mathrm{dx},} \quad b^{-1} \equiv \beta_N(\gamma - 1)q_N = O(1)$$

$$\int_0^\infty \Omega(\overline{\Theta}_N, \mathrm{x})\mathrm{dx} = 1$$

(Clavin He 1996)

Oscillatory instability



Stability analysis

$$\Theta_N(\mathrm{t}) = \overline{\Theta}_N + \delta\Theta_N(\mathrm{t})$$

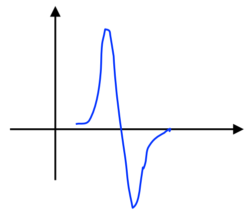
$\Theta_N \equiv \beta_N(T_N - \overline{T}_N)/\overline{T}_N$

$\overline{\Theta}_N = 0$

Linear integral equation : $\boxed{\delta\Theta_N(\mathrm{t}) = b^{-1}\int_0^\infty \Omega'_N(\mathrm{x})\delta\Theta_N(\mathrm{t}-\mathrm{x})\mathrm{dx},}$

$$\Omega'_N(\mathrm{x}) \equiv \partial\Omega/\partial\Theta_N|_{\Theta_N=\overline{\Theta}_N}$$

2 quantities:

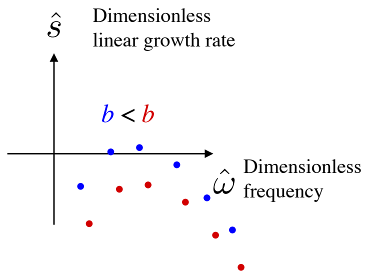


a function $\Omega'_N(\mathrm{x})$ $\qquad \int_0^\infty \Omega(\Theta_N, \mathrm{x})\mathrm{d}x = 1 \;\; \forall\Theta_N \;\; \Rightarrow \;\; \int_0^\infty \Omega'_N(\mathrm{x})\mathrm{dx} = 0$

a non-dimensional parameter of order unity: $b^{-1} \equiv \beta_N(\gamma - 1)q_N$

Complex eigenmodes

$$\delta\Theta_N(\mathrm{t}) \;\propto\; \exp(\hat{\sigma}\mathrm{t}) \qquad \hat{\sigma} = \hat{s} + \mathrm{i}\hat{\omega} \quad \text{(complex number)}$$



integral equation $\qquad b = \int_0^\infty \Omega'_N(\mathrm{x})\mathrm{e}^{-\hat{\sigma}\mathrm{x}}\mathrm{dx}$

$\hat{s} > 0$ and $\hat{\omega} \neq 0$: oscillatory instability $\qquad \hat{\omega} = O(1) \Leftrightarrow \omega = O(\bar{t}_N)$

$\hat{s} = 0$ and $\hat{\omega} \neq 0$: stability limit

Poincaré-Andronov (Hopf) bifurcation

frequence of oscillation
of order of the transit time

Discrete set of eigenmodes

The oscillatory instability is promoted by an increase of
- the thermal sensitivity β_N
- the heat release q_N
- the stiffness of the spatial distribution of heat release, function $\Omega'_N(\mathrm{x})$

The dynamics of the square-wave model is singular $\quad \Omega(\Theta_N, \mathrm{x}) = \delta(\mathrm{x} - l_N(\mathrm{t})), \quad l_N = \exp^{-\Theta_N} \quad \mathrm{d}\Theta_N(\mathrm{t}-1)/\mathrm{dt} = b\Theta_N(\mathrm{t})$

$$\hat{\sigma}\mathrm{e}^{-\hat{\sigma}} = b \qquad \hat{s}_i \to \infty, \quad \hat{\omega}_i \to \infty \qquad \forall\, b$$

Nonlinear dynamics: limit cycle, period doubling, chaos, dynamical quenching (Clavin He 1996)

$$1 + b\Theta_N(\mathrm{t}) = \int_0^\infty \Omega(\Theta_N(\mathrm{t}-\mathrm{x}), \mathrm{x})\mathrm{dx},$$

XII-4)CJ detonations for small heat release

(Clavin Williams 2002)

Reactive Euler equations in 1-D geometry

$D/Dt \equiv \partial/\partial t + \mathbf{u}.\nabla$

$\frac{1}{a\rho}\times$

$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla.\mathbf{u}, \quad \left(\rho\frac{D\mathbf{u}}{Dt} = -\nabla p,\right) \quad p = (c_p - c_v)\rho T,$

$a^2 = \gamma p/\rho \qquad$ 1-D : $\; D^{\pm}/Dt \equiv \partial/\partial t \pm (a \pm u)\partial/\partial x$

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}} \quad \boxed{\frac{D\psi}{Dt} = \frac{\dot{w}}{t_N},} \quad \boxed{\frac{1}{T}\frac{DT}{Dt} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{Dp}{Dt} = \frac{q_m}{c_pT}\frac{\dot{w}}{t_N},} \Rightarrow \pm\left(\frac{1}{\gamma p}\frac{Dp}{Dt} + \nabla.\mathbf{u} = \frac{q_m}{c_pT}\frac{\dot{w}}{t_N}\right)$$

$$\Rightarrow \frac{1}{\gamma p}\frac{D^{\pm}p}{Dt} \pm \frac{1}{a}\frac{D^{\pm}u}{Dt} = \frac{q_m}{c_pT}\frac{\dot{w}}{t_N}$$

entropy equation

$\dot{w}(\psi, T)$

1-D Euler (compressible) eqs.

$$\boxed{\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]p \pm \frac{1}{a}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{w}}{t_N}}$$

generalized acoustic eqs.
$(\delta p = \pm\rho a\delta u)$

(useful form for the following)

Near CJ regimes for small heat release. Transonic reacting flows

CJ and overdriven regimes

$$M_{u_{CJ}} = \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q}+1} \qquad \mathcal{Q} \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_pT_u} \qquad f \equiv \frac{\left(\overline{M}_u - \overline{M}_u^{-1}\right)^2}{4\mathcal{Q}}$$

CJ regime: $f = 1$, overdriven regime: $f > 1$

Small heat release approximation (transonic regimes)

$$\mathcal{Q} \ll 1$$

small parameter: $\boxed{\epsilon^2 \equiv \mathcal{Q} \ll 1}$ $\quad \boxed{M^2_{u_{CJ}} \approx 1 + 2\epsilon}$ $\quad$ overdriven regime near CJ: $\boxed{f = O(1)}$

$$\overline{M}_u^2 \approx 1 + 2\epsilon\sqrt{f} \qquad \overline{M}_u^2 - M^2_{u_{CJ}} \approx 2\epsilon(\sqrt{f} - 1)$$

⚠ ϵ in p.9 $\neq \epsilon$ in p.5

$$q_N \equiv q_m / c_p T_N = O(\epsilon^2) \qquad f = O(1)$$

Rankine-Hugoniot conditions $\quad (M_u^2 - 1) \ll 1$

see p.6 lecture X

$$\frac{T_N}{T_u} \approx 1 + (\gamma - 1)(M_u^2 - 1), \qquad \frac{p_N}{p_u} \approx \frac{\rho_N}{\rho_u} \approx 1 + (M_u^2 - 1), \qquad \psi = 0$$

Steady state $\quad (\overline{M}_u^2 - 1) \approx (1 - \overline{M}_N^2) \approx 2\epsilon\sqrt{f}$

heat release

compressible effect

$$\frac{\overline{T} - \overline{T}_N}{T_u} \approx \epsilon^2 \psi - (\gamma - 1)\epsilon\sqrt{f}[1 - \sqrt{1 - (\psi/f)}], \qquad (1 - \overline{M}_b^2) \approx 2\epsilon\sqrt{f - 1}$$

$$\frac{(\gamma + 1)}{2\gamma} \frac{(\overline{p} - p_u)}{p_u} = \frac{(\gamma + 1)}{2} \overline{M}_u^2 \frac{(\overline{\mathcal{D}} - \overline{u})}{\overline{\mathcal{D}}} \approx \epsilon\sqrt{f}\,\overline{M}_u[1 + \sqrt{1 - (\psi/f)}],$$

! crossover temperature : $\quad T_u < T^* < T_N$ $\qquad$ separation of scale: $\tau_{coll}/\bar{t}_N \ll (\overline{M}_u - 1) \Rightarrow \mathrm{e}^{-E/k_B T_N} \ll \epsilon$

Distinguished limit $\quad \boxed{(\gamma - 1) = O(\epsilon)} \Rightarrow (T - T_N)/T_N = O(\epsilon^2)$

Non-dimensional equations $\quad \mathrm{t} \equiv \dfrac{t}{\bar{t}_N}, \quad \mathrm{x} \equiv \dfrac{x}{a_u \bar{t}_N}, \quad \breve{u} \equiv \dfrac{u}{a_u}, \quad \breve{\pi} \equiv \dfrac{1}{\gamma} \ln\left(\dfrac{p}{p_u}\right), \quad \breve{\theta} \equiv \dfrac{(T - \hat{T}_u)}{\hat{T}_u}$

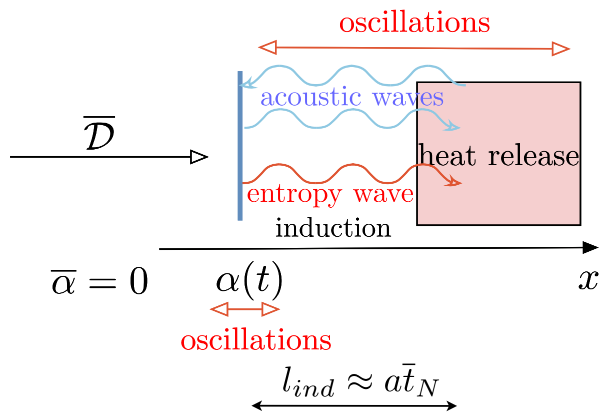
anticipating $\quad \breve{\theta} = O(\epsilon^2) \quad a/a_u = 1 + O(\epsilon^2) \quad 1 - \breve{u} = O(\epsilon) \quad \breve{\pi} = O(\epsilon)$

the variation of a is negligible in $\dfrac{1}{\gamma p}\left[\dfrac{\partial}{\partial t} + (u \pm a)\dfrac{\partial}{\partial x}\right] p \pm \dfrac{1}{a}\left[\dfrac{\partial}{\partial t} + (u \pm a)\dfrac{\partial}{\partial x}\right] u = \dfrac{q_m}{c_p T} \dfrac{\dot{\mathrm{w}}}{\bar{t}_N}$

$$\boxed{\begin{aligned} &\left[\frac{\partial}{\partial \mathrm{t}} + (1 + \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2 \dot{\mathrm{w}}, && \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma - 1)\breve{\pi}] = \epsilon^2 \dot{\mathrm{w}} \\ &\left[\frac{\partial}{\partial \mathrm{t}} - (1 - \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} - \breve{u}) = \epsilon^2 \dot{\mathrm{w}}, && \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}}, \end{aligned}}$$

Slow time scale

$M_u^2 - 1 = O(\epsilon) \Rightarrow$ transonic flow: $u/a = 1 + O(\epsilon)$



time scale of the downward propagating acoustic wave : $l_{ind}/a = \bar{t}_N$

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]p + \frac{1}{a}\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

time scale of the upward propagating acoustic wave : $l_{ind}/(a-u) \approx \bar{t}_N/\epsilon$

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right]p - \frac{1}{a}\left[\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

longest delay in the feed back loop $\approx \bar{t}_N/\epsilon$

Scaling

$$\boxed{\text{Period of oscillation} = O(\bar{t}_N/\epsilon)}$$

non dimensional time of order unity $\boxed{\tau \equiv \dfrac{t}{\bar{t}_N/\epsilon}} = \epsilon\,\mathrm{t} \quad \leftarrow \quad \mathrm{t} \equiv t/\bar{t}_N$

instantaneous position of the lead shock wave $x = \alpha(\epsilon\, t/\bar{t}_N)$ $\quad \mathrm{a} \equiv \alpha(\tau)/(a_u \bar{t}_N) \leftarrow$ non dimensional position

non dimensional variable of order unity $\boxed{\mu,\ \pi,\ \theta :}$ $\quad \breve{u} \equiv \dfrac{u}{a_u} = 1 + \epsilon\mu, \quad \breve{\pi} \equiv \dfrac{1}{\gamma}\ln\left(\dfrac{\hat{p}}{\hat{p}_u}\right) = \epsilon\pi, \quad \breve{\theta} \equiv \dfrac{(T - \hat{T}_u)}{T_u} = \epsilon^2\theta$

reference frame of the moving shock : $\quad \tau \equiv \epsilon \mathrm{t}, \quad \xi \equiv \mathrm{x} - \mathrm{a}(\tau), \quad \dfrac{\partial}{\partial \mathrm{x}} = \dfrac{\partial}{\partial \xi}, \quad \dfrac{\partial}{\partial \mathrm{t}} = \epsilon\left(\dfrac{\partial}{\partial \tau} - \dot{\mathrm{a}}_\tau \dfrac{\partial}{\partial \xi}\right)$

! u and μ are flow velocities in the lab frame $\quad \mathrm{x} \equiv x/(a_u \bar{t}_N)$

$$\left[\frac{\partial}{\partial \mathrm{t}} + (1+\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2 \dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma-1)\breve{\pi}] = \epsilon^2 \dot{\mathrm{w}}$$

$$\left[\frac{\partial}{\partial \mathrm{t}} - (1-\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} - \breve{u}) = \epsilon^2 \dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}},$$

$$\Rightarrow \quad \boxed{\begin{aligned} &\frac{\partial(\pi+\mu)}{\partial \xi} = 0 && \frac{\partial(\theta - h\pi - \psi)}{\partial \xi} = 0, \\ &\left[\frac{\partial}{\partial \tau} + (\mu - \dot{\mathrm{a}}_\tau)\frac{\partial}{\partial \xi}\right](\pi - \mu) = \dot{\mathrm{w}}, && \frac{\partial \psi}{\partial \xi} = \dot{\mathrm{w}} \end{aligned}}$$

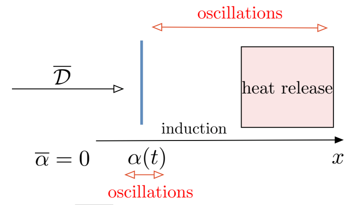
$h \equiv (\gamma - 1)/\epsilon = O(1)$

$\dot{\mathrm{a}}_\tau \equiv \mathrm{da}(\tau)/\mathrm{d}\tau$

Arrhenius law : $\quad \dot{\mathrm{w}}(\psi,\theta) = (1-\psi)\mathrm{e}^{\beta_e(\theta - \overline{\theta}_N)}$ with $\boxed{\beta_e \equiv \dfrac{E}{k_B T_N}\epsilon^2 = O(1)}$

Asymptotic model for CJ or near CJ regimes

$\epsilon^2 \equiv q_m/c_p T_u \ll 1, \quad (\gamma-1) = O(\epsilon), \quad E/k_B T_N = O(1/\epsilon^2)$ (Clavin Williams 2002)



$$\frac{\partial(\pi+\mu)}{\partial\xi} = 0 \qquad \frac{\partial(\theta - h\pi - \psi)}{\partial\xi} = 0,$$

$$\left[\frac{\partial}{\partial\tau} + (\mu - \dot{\mathrm{a}}_\tau)\frac{\partial}{\partial\xi}\right](\pi - \mu) = \dot{\mathrm{w}}, \qquad \frac{\partial\psi}{\partial\xi} = \dot{\mathrm{w}}$$

$$\dot{\mathrm{a}}_\tau \equiv \mathrm{da}(\tau)/\mathrm{d}\tau$$

$$M_u = (\overline{\mathcal{D}} - \dot{\alpha}_t)/a_u = \overline{M}_u - \epsilon\,\dot{\mathrm{a}}_\tau$$

Boundary conditions at the Neumann state

$h \equiv (\gamma-1)/\epsilon = O(1) \qquad f \equiv$ overdrive factor

$$\frac{T_N}{T_u} \approx 1 + (\gamma-1)(M_u^2 - 1),$$

$$\frac{p_N}{p_u} \approx \frac{\rho_N}{\rho_u} \approx 1 + (M_u^2 - 1)$$

$$M_u^2 - 1 \approx 2(\overline{M}_u - 1) - 2\epsilon\,\dot{\mathrm{a}}_\tau$$

$$(\overline{M}_u - 1) \approx \varepsilon\sqrt{f}$$

$$\rho_u(\overline{\mathcal{D}} - \dot{\alpha}_t) = \rho_N(u|_{x=\alpha} - \dot{\alpha}_t)$$

⚠ u and μ are flow velocities in the lab frame

$$\xi = 0: \quad \theta \equiv \theta_N = 2h(\sqrt{f} - \dot{\mathrm{a}}_\tau), \quad \pi \equiv \pi_N = 2(\sqrt{f} - \dot{\mathrm{a}}_\tau)$$

$$\theta_N - h\pi_N = 0$$

$$\xi = 0: \quad \mu \equiv \mu_N = -\sqrt{f} + 2\dot{\mathrm{a}}_\tau \quad \text{and} \quad \psi = 0,$$

$$\mu_N + \pi_N = \sqrt{f}$$

$$\mu + \pi = \sqrt{f} \qquad \theta = h\sqrt{f} - h\mu + \psi$$

The problem is reduced to solve two equations for μ and ψ

$$\boxed{\begin{array}{l} \left[\frac{\partial}{\partial\tau} + (\mu - \dot{\mathrm{a}}_\tau)\frac{\partial}{\partial\xi}\right]\mu = -\frac{\dot{\mathrm{w}}}{2}, \qquad \frac{\partial\psi}{\partial\xi} = \dot{\mathrm{w}}(\psi, \theta) \\ \xi = 0: \quad \mu = -\sqrt{f} + 2\,\dot{\mathrm{a}}_\tau \quad \text{and} \quad \psi = 0, \end{array}}$$

Boundary condition in the burnt gas

$$\boxed{\xi \to \infty: \qquad \psi = 1, \qquad \mu = \overline{\mu}_b = -\sqrt{f-1}}$$

yields an integral equation for $\dot{\mathrm{a}}_\tau(\tau)$

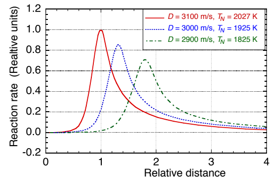
Nonlinear equation for a transonic reacting flow

$$\left[\frac{\partial}{\partial\tau} + (\mu - \dot{a}_\tau)\frac{\partial}{\partial\xi}\right]\mu = -\frac{\dot{w}}{2}, \qquad \frac{\partial\psi}{\partial\xi} = \dot{w}(\psi,\theta) \qquad \dot{w}(\psi=1)=0$$

$$\theta = h\sqrt{f} - h\mu + \psi$$

$$\xi = 0: \quad \mu = -\sqrt{f} + 2\,\dot{a}_\tau \quad \text{and} \quad \psi = 0, \qquad \xi \to \infty: \quad \psi = 1, \quad \mu = \overline{\mu}_b = -\sqrt{f-1}$$

Result for simplified chemical kinetics



Simplification:

The reaction rate depends only on $T_N(t)$

The stability analysis is similar to that of strongly overdriven regimes !

Similar integral equation but with a delay controlled by the upstream running acoustic wave

$$\Delta(\xi) = \int_0^\xi \frac{\mathrm{d}\xi}{|\mu(\xi)|}$$

$$\boxed{\dot{a}_\tau(\tau) = \int_0^\infty \left[\frac{1}{4\sqrt{f}}\Omega'_N(\xi) + G(\xi)\right]\dot{a}_\tau(\tau - \Delta(\xi))\mathrm{d}\xi}$$

Instability due to thermal sensitivity

Stabilizing term due to residual compressible effects

Conclusion

> Galloping detonations are due to a phase shift in the loop between the lead shock and the heat release, controlled by the entropy wave and the upstream running acoustic wave