Tsinghua-Princeton-CI Summer School
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Lectures on Dynamics of Combustion Waves in Premixed Gases

Professor Paul Clavin

Aix-Marseille Université
ECM & CNRS (IRPHE)

Lecture XIII
Stability analysis of shock waves

Lecture 13 : Stability analysis of shock waves

13-1. Acoustic waves and entropy-vorticity wave

Linearized Euler equations

Linearized flow field

13-2. Analyses

Dispersion relation for general materials

Classification of normal modes

Spontaneous emission of sound and instability

Stability of shocks in ideal gases

Stability of reacting shocks

XIII-1) Acoustic waves and entropy-vorticity wave

Shock wave $\approx$ hydrodynamic discontinuity + Rankine-Hugoniot conditions

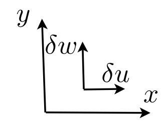
The flow of shocked gas in a planar wave is uniform

$\mathcal{D} > a_u \Rightarrow$ the upstream flow in a wrinkled shock is not perturbed

Flow velocity $\overline{u}_N$ is sufficiently large $\Rightarrow$ the diffusive fluxes are negligible: $\mathrm{D}s/\mathrm{D}t = 0$

The entropy of shocked gas is modified at the Neumann state of a wrinkled shock $\Rightarrow \nabla s(\mathbf{r}, t) \neq 0$

Linearized Euler equations



(written in 2-D for simplicity. Extension to 2-D is straightforward)

$$u = \overline{u}_N + \delta u, \quad w = \delta w, \quad \rho = \overline{\rho}_N + \delta\rho, \quad p = \overline{p}_N + \delta p$$

$$\frac{1}{\overline{\rho}_N}\frac{\mathrm{D}}{\mathrm{D}t}\delta\rho + \frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta w = 0,$$

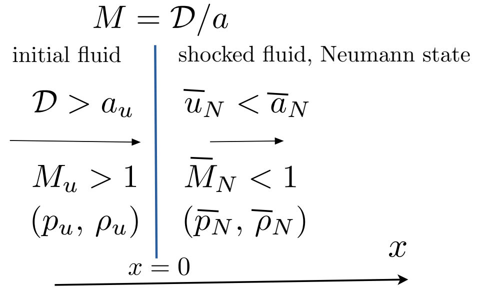
$\mathrm{D}/\mathrm{D}t = \partial/\partial t + \overline{u}_N \partial/\partial x$

$$\overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta u = -\frac{\partial}{\partial x}\delta p, \qquad \overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta w = -\frac{\partial}{\partial y}\delta p,$$

$\partial p/\partial\rho|_{s=\mathrm{cst}} \equiv a$

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta s = 0 \Rightarrow \frac{\mathrm{D}}{\mathrm{D}t}\delta p = \overline{a}_N^2 \frac{\mathrm{D}}{\mathrm{D}t}\delta\rho,$$

in the frame of the unperturbed planar shock



Wave equation for the pressure (d'Alembert equation)

eliminating $\delta\rho \quad \Rightarrow \quad \dfrac{1}{\overline{\rho}_N \overline{a}_N^2}\dfrac{\mathrm{D}}{\mathrm{D}t}\delta p + \dfrac{\partial}{\partial x}\delta u + \dfrac{\partial}{\partial y}\delta w = 0$

eliminating δu and $\delta w \quad \Rightarrow \quad \boxed{\dfrac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p - \overline{a}_N^2\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\delta p = 0}$

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity $\overline{u}_N$

Linearized flow field

Flow splitting

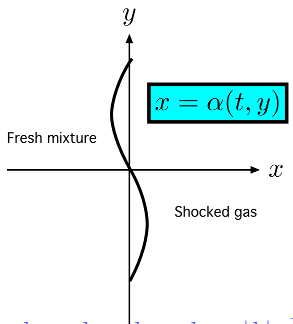
acoustic wave + vorticity wave

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{u}_N \frac{\partial}{\partial x}$$

$$\delta p = \delta p^{(a)}, \quad \boxed{\begin{aligned}\delta u &= \delta u^{(a)} + \delta u^{(i)} \\ \delta w &= \delta w^{(a)} + \delta w^{(i)}\end{aligned}} \qquad \begin{aligned}\mathrm{D}\delta u^{(i)}/\mathrm{D}t = 0 \\ \mathrm{D}\delta w^{(i)}/\mathrm{D}t = 0\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}\frac{\partial \delta u^{(i)}}{\partial t} + \overline{u}_N \frac{\partial \delta u^{(i)}}{\partial x} = 0 \\ \frac{\partial \delta w^{(i)}}{\partial t} + \overline{u}_N \frac{\partial \delta w^{(i)}}{\partial x} = 0\end{aligned} \quad \Leftrightarrow \quad \boxed{\begin{aligned}\delta u^{(i)}(y, t - x/\overline{u}_N) \\ \delta w^{(i)}(y, t - x/\overline{u}_N)\end{aligned}}$$

$$\frac{1}{\overline{\rho}_N \overline{a}_N^2} \cancel{\frac{\mathrm{D}}{\mathrm{D}t}} \delta p + \frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta w = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}\delta u^{(i)} + \frac{\partial}{\partial y}\delta w^{(i)} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial t}\delta u^{(i)} = \overline{u}_N \frac{\partial}{\partial y}\delta w^{(i)}}$$

Normal-mode analysis



$$\alpha(y,t) = \hat{\alpha} e^{iky+\sigma t} \qquad \boxed{\delta f(x,y,t) = \tilde{f}(x) e^{iky+\sigma t}} \qquad \begin{aligned} k \in \mathrm{Re} \\ \sigma \in \mathrm{Z}\end{aligned} \quad \boxed{\sigma(k)\ ?}$$

$$\delta p = \tilde{p}_N \exp\left(\mathrm{i}l_\pm x + \mathrm{i}ky + \sigma t\right) \qquad x = 0: \ \delta p = \delta p_N(y,t) = \tilde{p}_N \mathrm{e}^{\mathrm{i}ky+\sigma t}$$

RH conditions

2nd-order algebraic eq.

$$l_\pm(\sigma, k) \qquad (\sigma + \mathrm{i}l_\pm \overline{u}_N)^2 + \overline{a}_N^2 (l_\pm^2 + k^2) = 0 \quad \Leftarrow \quad \frac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p - \overline{a}_N^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta p = 0 \quad \text{wave eq.}$$

No length-scale other than $|k|^{-1}$ in the problem

$$\mathrm{i}\frac{l_\pm}{|k|} = \frac{\overline{M}_N S \pm \sqrt{1+S^2}}{\sqrt{1-\overline{M}_N^2}} \quad \text{with} \quad S \equiv \frac{\sigma}{\overline{a}_N |k|} \frac{1}{\sqrt{1-\overline{M}_N^2}},$$

$\delta u^{(i)}(y, t - x/\overline{u}_N)$
$\delta w^{(i)}(y, t - x/\overline{u}_N)$

$$\begin{aligned}\overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta u = -\frac{\partial}{\partial x}\delta p, \\ \overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta w = -\frac{\partial}{\partial y}\delta p,\end{aligned} \quad \Rightarrow \quad \begin{aligned}\tilde{u}^{(a)} &= -\frac{\mathrm{i}l_\pm \overline{u}_N}{\sigma + \mathrm{i}l_\pm \overline{u}_N} \frac{\tilde{p}_N}{\overline{\rho}_N \overline{u}_N} \mathrm{e}^{\mathrm{i}l_\pm x}, \qquad & \tilde{u}^{(i)} &= \left[\tilde{u}_N + \frac{\mathrm{i}l_\pm \overline{u}_N}{\sigma + \mathrm{i}l_\pm \overline{u}_N} \frac{\tilde{p}_N}{\overline{\rho}_N \overline{u}_N}\right] \mathrm{e}^{-\sigma x/\overline{u}_N}, \\ \tilde{w}^{(a)} &= -\frac{\mathrm{i}k \overline{u}_N}{\sigma + \mathrm{i}l_\pm \overline{u}_N} \frac{\tilde{p}_N}{\overline{\rho}_N \overline{u}_N} \mathrm{e}^{\mathrm{i}l_\pm x}, & \tilde{w}^{(i)} &= \left[\tilde{w}_N + \frac{\mathrm{i}k \overline{u}_N}{\sigma + \mathrm{i}l_\pm \overline{u}_N} \frac{\tilde{p}_N}{\overline{\rho}_N \overline{u}_N}\right] \mathrm{e}^{-\sigma x/\overline{u}_N},\end{aligned}$$

Neumann state　　　Neumann state

Incompressibility condition

$$\frac{\partial}{\partial t}\delta u^{(i)} = \overline{u}_N \frac{\partial}{\partial y}\delta w^{(i)} \quad \Rightarrow \quad \boxed{-\frac{(\mathrm{i}l_\pm \sigma + \overline{u}_N k^2)}{(\sigma + \mathrm{i}l_\pm \overline{u}_N)} \frac{\tilde{p}_N}{\overline{\rho}_N \overline{u}_N} - \sigma \frac{\tilde{u}_N}{\overline{u}_N} + \mathrm{i}k\tilde{w}_N = 0}$$

XIII-2 Analyses

Dispersion relation for general materials

Compatibility condition

$$-\frac{(\mathrm{i}l_\pm\sigma+\bar u_N k^2)}{\cancel{(\sigma+\mathrm{i}l_\pm\bar u_N)}}\frac{\tilde p_N}{\bar\rho_N\bar u_N}-\sigma\frac{\tilde u_N}{\bar u_N}+\mathrm{i}k\tilde w_N=0$$

$$\left(\frac{\partial}{\partial t}+\bar u_N\frac{\partial}{\partial x}\right)^2\delta p-\bar a_N^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta p=0 \qquad (\sigma+\mathrm{i}l_\pm\bar u_N)^2+\bar a_N^2(l_\pm^2+k^2)=0 \;\Rightarrow\; \sigma^2+2\mathrm{i}\sigma l_\pm\bar u_N-l_\pm^2\bar u^2+\bar a_N^2(l_\pm^2+k^2)=0$$

$$-(\mathrm{i}l_\pm\bar u_N\sigma+\bar u_N^2k^2)=\sigma^2+\mathrm{i}l_\pm\sigma\bar u_N+(\bar a_N^2-\bar u_N^2)(l_\pm^2+k^2)$$

$$\boxed{\mathrm{i}\frac{l_\pm}{|k|}=\frac{\bar M_N S\pm\sqrt{1+S^2}}{\sqrt{1-\bar M_N^2}}} \qquad \bar a_N^2(l_\pm^2+k^2)=-(\sigma+\mathrm{i}l_\pm\bar u_N)^2 \quad \Rightarrow \quad =(\sigma+\mathrm{i}l_\pm\bar u_N)\left[\sigma-\left(1-\bar M_N^2\right)(\sigma+\mathrm{i}l_\pm\bar u_N)\right]$$

$$\text{with}\quad S\equiv\frac{\sigma}{\bar a_N|k|}\frac{1}{\sqrt{1-\bar M_N^2}} \quad \Rightarrow \quad =(\sigma+\mathrm{i}l_\pm\bar u_N)\left[\sigma\bar M_N^2-\left(1-\bar M_N^2\right)\mathrm{i}l_\pm\bar u_N\right]$$

$$-(\mathrm{i}l_\pm\bar u_N\sigma+\bar u_N^2k^2)=-\cancel{(\sigma+\mathrm{i}l_\pm\bar u_N)}\sqrt{1-\bar M_N^2}\left[\pm\sqrt{1+S^2}\right]|k|\bar u_N$$

$$-\frac{(\mathrm{i}l_\pm\sigma+\bar u_N k^2)}{(\sigma+\mathrm{i}l_\pm\bar u_N)}\frac{\tilde p_N}{\bar\rho_N\bar u_N}-\sigma\frac{\tilde u_N}{\bar u_N}+\mathrm{i}k\tilde w_N=0 \;\Rightarrow\; -\sqrt{1-\bar M_N^2}\left[\pm\sqrt{1+S^2}\right]|k|\bar u_N\frac{\tilde p_N}{\bar\rho_N\bar u_N}-\sigma\frac{\tilde u_N}{\bar u_N}+\mathrm{i}k\tilde w_N=0$$

$$\pm\sqrt{S^2+1}\boxed{\frac{\tilde p_N}{\bar\rho_N\bar a_N\bar u_N}}+S\boxed{\frac{\tilde u_N}{\bar u_N}}-\frac{\mathrm{i}k}{|k|}\boxed{\frac{\tilde w_N}{\bar u_N}}\frac{\bar M_N}{\sqrt{1-\bar M_N^2}}=0$$

(Buckmaster Ludford 1988. Clavin et al. 1997)

The Rankine Hugoniot relations yields an equation for $S\propto\dfrac{\sigma}{\bar a_N|k|}$

$$\frac{\delta p_N}{\bar p_N}\propto\frac{\dot\alpha_t}{\bar u_N},\quad \boxed{\frac{\tilde p_N}{\bar p_N}\propto \mathrm{i}\sigma\frac{\hat\alpha}{\bar u_N}} \qquad \frac{\delta u_N}{\bar u_N}\propto\frac{\dot\alpha_t}{\bar u_N},\quad \boxed{\frac{\tilde u_N}{\bar u_N}\propto \mathrm{i}\sigma\frac{\hat\alpha}{\bar u_N}} \qquad \frac{\delta w_N}{\bar u_N}\propto\alpha'_y,\quad \boxed{\frac{\tilde w_N}{\bar u_N}\propto \mathrm{i}k\hat\alpha}$$

Downstream boundary condition

$x\rightarrow\infty$: bounded condition (in the unstable case, $\mathrm{Re}\sigma>0$)

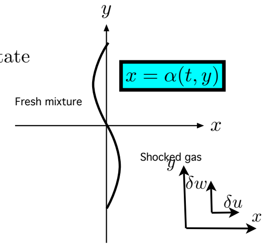
$$\delta p=\tilde p_N\exp\left(\mathrm{i}l_\pm x+\mathrm{i}ky+\sigma t\right)$$

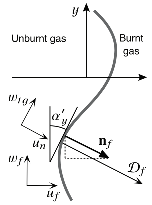
$\boxed{\text{selection of the the sign in } l_\pm \text{ such that } e^{il_\pm x} \text{ does not diverge}}$

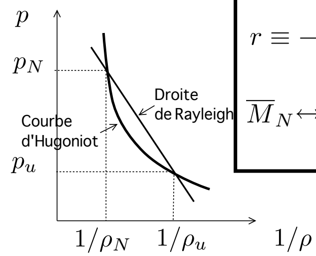
Rankine Hugoniot relations (general material)

$M = \mathcal{D}/a$

initial fluid	shocked fluid, Neumann state
$\mathcal{D} > a_u$	$u_N < a_N$
$M_u > 1$	$M_N < 1$
$(p_u,\ \rho_u)$	$(p_N,\ \rho_N)$







non-dimensional parameters

$$r \equiv -\frac{(\rho_u \mathcal{D})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0, \qquad \overline{M}_N \leftrightarrow n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)}$$

$$\boxed{\delta p_N = \frac{1}{r}\left(\frac{\rho_u}{\overline{\rho}_N}\right)^2 \overline{\mathcal{D}}^2 \delta\rho_N}$$

Jump conditions

p.5 lecture IV

mass $\quad \rho_N(u_N - \partial\alpha/\partial t - w_N \partial\alpha/\partial y) = \rho_u(\overline{\mathcal{D}} - \partial\alpha/\partial t) \quad \Rightarrow \quad \delta\rho_N \overline{u}_N + \overline{\rho}_N(\delta u_N - \partial\alpha/\partial t) = -\rho_u \partial\alpha/\partial t$

$$\boxed{\delta m = -\rho_u \frac{\partial\alpha}{\partial t}}$$

tangential momentum $\quad w_N = (\overline{\mathcal{D}} - u_N)\alpha'_y \quad \Rightarrow \quad \delta w_N = (\overline{\mathcal{D}} - \overline{u}_N)\partial\alpha/\partial y$

tangent to the Hugoniot curve (geometrical construction) $\quad \Rightarrow \quad \dfrac{\delta p_N}{\overline{p}_N} = \dfrac{1}{r}\dfrac{(\rho_u \mathcal{D})^2}{\overline{p}_N \overline{\rho}_N}\dfrac{\delta\rho_N}{\overline{\rho}_N}$

longitudinal momentum

$$\left.\begin{array}{l} p_N - p_u = -m^2\left(\dfrac{1}{\rho_N} - \dfrac{1}{\rho_u}\right) \\ \dfrac{\delta p_N}{\overline{p}_N} = \dfrac{2}{\overline{m}}\left(1 - \dfrac{p_u}{\overline{p}_N}\right)\rho_u \dfrac{\partial\alpha}{\partial t} + \dfrac{\overline{m}^2}{\overline{p}_N \overline{\rho}_N}\dfrac{\delta\rho_N}{\overline{\rho}_N} \end{array}\right\} \Rightarrow$$

$$\frac{\delta p_N}{\overline{p}_N} = -2\frac{\left(1 - \frac{p_u}{\overline{p}_N}\right)}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}, \qquad \frac{\delta\rho_N}{\overline{\rho}_N} = -2\left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{r}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}$$

$$\frac{\delta u_N}{\overline{u}_N} = \left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{1+r}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}, \qquad \frac{\delta w_N}{\overline{u}_N} = \left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{\partial\alpha}{\partial y}$$

Linear rate

$$\pm\sqrt{S^2+1}\frac{\tilde{p}_N}{\overline{\rho}_N \overline{a}_N \overline{u}_N} = -S\frac{\tilde{u}_N}{\overline{u}_N} + \frac{\mathrm{i}k\tilde{w}_N}{|k|\overline{u}_N}\frac{\overline{M}_N}{\sqrt{1-\overline{M}_N^2}}$$

$$\alpha(y,t) = \hat{\alpha}\mathrm{e}^{\mathrm{i}ky+\sigma t} \qquad \frac{\partial\alpha}{\partial t} = \sigma\alpha \qquad \frac{\partial\alpha}{\partial y} = \mathrm{i}k\alpha$$

$$\Rightarrow \quad \boxed{\begin{array}{c} \pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n \\ S \equiv \dfrac{\sigma}{\overline{a}_N|k|}\dfrac{1}{\sqrt{1-\overline{M}_N^2}} \end{array}}$$

taking the square $\quad \boxed{aS^4 + 2bS^2 + c = 0,} \qquad S^2 \equiv \dfrac{\sigma^2}{\overline{a}_N^2 k^2}\dfrac{1}{(1-M_N^2)}$

$$a \equiv (1+r)^2 - 4\overline{M}_N^2, \quad b \equiv (1-r^2)n - 2\overline{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

Quadratic equation for $\sigma^2/\overline{a}_N^2 k^2$

Clavin Williams 2012

Classification of normal modes

Normal modes for general materials

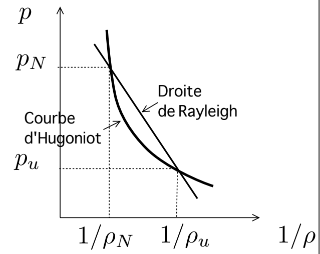
$$\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n$$

$$\boxed{aS^4 + 2bS^2 + c = 0} \qquad S \equiv \frac{\sigma}{\overline{a}_N|k|}\frac{1}{\sqrt{1-\overline{M}_N^2}}$$

$$a \equiv (1+r)^2 - 4\overline{M}_N^2, \quad b \equiv (1-r^2)n - 2\overline{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

2 non-dimensional parameters

$$r \equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0, \qquad n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)}$$



only the roots that satisfy boundedness condition should be retained $\quad \mathrm{Re}(il_\pm) \leqslant 0, \quad \mathrm{Re}\left(\overline{M}_N S \pm \sqrt{1+S^2}\right) \leqslant 0$

$\mathrm{Re}(\sigma) < 0$: stable mode exponentially damped

$\mathrm{Re}(\sigma) > 0$: unstable mode exponentially amplified

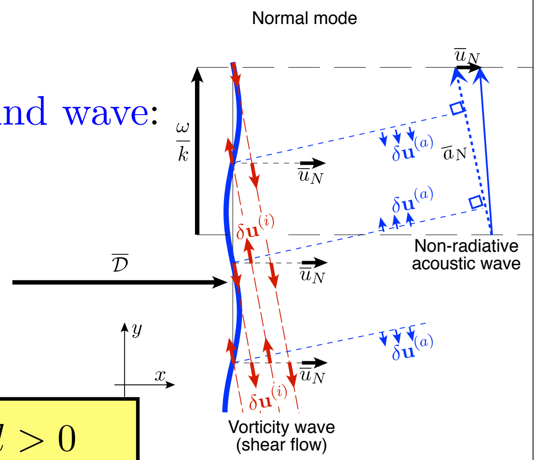
$S^2 < 0$: $\quad \mathrm{Re}(\sigma) = 0, \quad \omega \equiv \mathrm{Im}(\sigma) \neq 0$ neutral oscillatory modes $\quad S = \pm i\Omega, \quad \Omega\sqrt{1-\overline{M}_N^2} \equiv \omega/(\overline{a}_N|k|) > 0 \quad \Omega > 1 \quad il_\pm = \pm il,$

$$\delta p = \tilde{p}_N \exp\left[\mathrm{i}\mathbf{K}.(\ \mathbf{r} - \overline{u}_N\ \mathbf{e}_x t + \overline{a}_N\ \mathbf{e}_K t)\right] \qquad \omega = \overline{a}_N\sqrt{l^2+k^2} - \overline{u}_N l \qquad (l/|k|)\sqrt{1-\overline{M}_N^2} = \overline{M}_N\Omega + \sqrt{\Omega^2-1} > 0$$

$$\mathbf{r} \equiv x\mathbf{e}_x + y\mathbf{e}_y \qquad \mathbf{K} \equiv l\mathbf{e}_x + k\mathbf{e}_y \qquad \mathbf{e}_K \equiv \mathbf{K}/|\mathbf{K}|$$

longitudinal component of the velocity (unperturbed shock) of the sound wave:

$$\boxed{\mathbf{e}_x.(\overline{u}_N\mathbf{e}_x - \overline{a}_N\mathbf{e}_K) = \overline{u}_N - \overline{a}_N\frac{l}{\sqrt{l^2+k^2}}}$$



Neutral oscillatory modes

$$\boxed{\begin{array}{r} \text{Spontaneous generation of sound. Radiating condition: } \overline{u}_N\sqrt{l^2+k^2} - \overline{a}_N l > 0 \\ \text{Non-radiating condition: } \overline{u}_N\sqrt{l^2+k^2} - \overline{a}_N l < 0 \end{array}}$$

Spontaneous emission of sound and instability

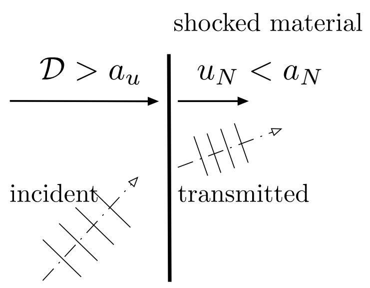
D'Yakov Kontorovich 1954-57

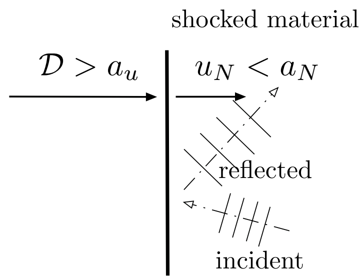
Oscillatory neutral modes

$\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n$
neutral oscillatory mode
$S = i\Omega, \quad \Omega > 1$

$$\Longrightarrow \begin{cases} \text{radiating waves:} \quad l/|k| = \left[\overline{M}_N\Omega - \sqrt{\Omega^2-1}\right]/\sqrt{1-\overline{M}_N^2}, /\sqrt{1-\overline{M}_N^2}, \\ 2\overline{M}_N\Omega\sqrt{\Omega^2-1} = -\Omega^2(1+r) + (1-r)n > 0, \\ \text{non-radiating waves:} \quad l/|k| = \left[\overline{M}_N\Omega + \sqrt{\Omega^2-1}\right]/\sqrt{1-\overline{M}_N^2}, \\ -2\overline{M}_N\Omega\sqrt{\Omega^2-1} = -\Omega^2(1+r) + (1-r)n < 0 \end{cases}$$

Transmitted or reflected sound wave





If a normal mode is radiating the response of the shock diverges when the reflected (or transmitted) waves matched the radiating normal mode

$$\begin{array}{l} \text{reflected} \longrightarrow \\ \text{incident} \longrightarrow \end{array} \frac{\tilde{p}_r}{\tilde{p}_i} = -\frac{\left[2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n\right]}{\left[-2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n\right]} \longleftarrow \text{denominator goes trough 0}$$

A neutral oscillatory mode that is radiating is considered as unstable

D'Yakov Kontorovich 1954-57

Power laws of neutral modes

Square root in the dispersion relation $\pm 2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n = 0$ $\Longrightarrow$ cut in the complex plane $\Longrightarrow$ power laws

Bates 2007

Damping ($n < 0$) or amplification ($n > 0$) involving power laws t^n
can be described by Laplace transform not by Fourier transform

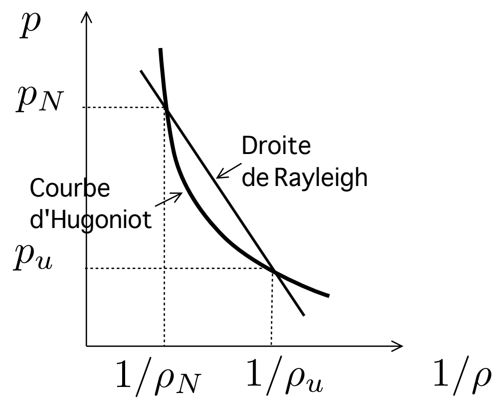
Neutral modes with non-radiating acoustic waves relaxe follc ing a power law in time $\quad t^n \quad n < 0$

Neutral modes with a radiating acoustic wave is unstable according a power law in time $\quad t^n \quad n > 0$

Classification of normal modes

2 non-dimensional parameters

$$\boxed{\begin{aligned} r &\equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0, \\ n &\equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)} \end{aligned}}$$

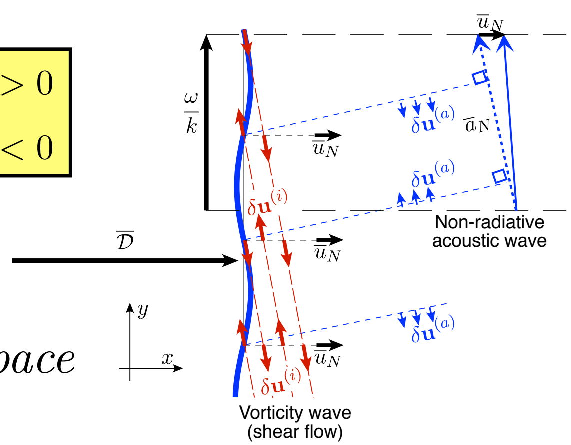


Neutral oscillatory modes

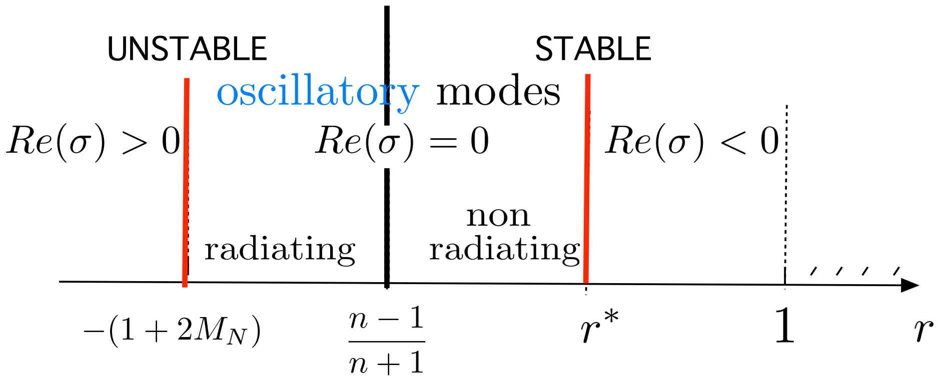
Spontaneous generation of sound. Radiating condition: $\overline{u}_N\sqrt{l^2+k^2}-\overline{a}_N l > 0$

Non-radiating condition: $\overline{u}_N\sqrt{l^2+k^2}-\overline{a}_N l < 0$

Normal mode



Classification of the normal modes in the parameters space



$$r^* = \frac{n-\sqrt{(1-M_N^2)\left(1-\frac{\rho_u}{\rho_N}\right)}}{n+1}$$

Clavin Williams 2012

Stability of shocks in ideal gases

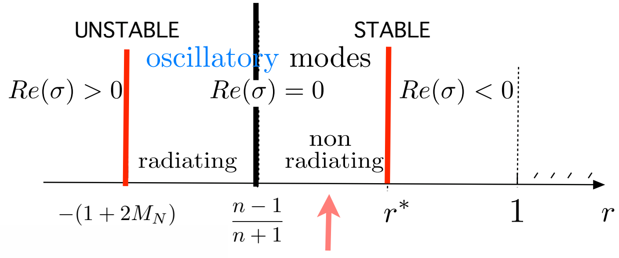
polytropic gas, $\gamma = \text{cst.}$ $\quad \dfrac{u_N}{\mathcal{D}} = \dfrac{\rho_u}{\rho_N} = \dfrac{(\gamma-1)M_u^2+2}{(\gamma+1)M_u^2}, \qquad \dfrac{p_N}{p_u} = \dfrac{2\gamma M_u^2-(\gamma-1)}{(\gamma+1)}, \qquad M_N^2 = \dfrac{(\gamma-1)M_u^2+2}{2\gamma M_u^2-(\gamma-1)}$

$$r \equiv -\frac{(\rho_u\overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} = \frac{1}{\overline{M}_u^2}, \qquad n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)} = \frac{\overline{M}_u^2}{(\overline{M}_u^2-1)}$$

$$\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n, \quad \Rightarrow \quad \pm 2S\overline{M}_N\sqrt{1+S^2} = S^2\left(1+\overline{M}_u^{-2}\right)+1$$

$$\overline{M}_u > 1, \quad \gamma > 1 \quad \Rightarrow \quad \boxed{(n-1)/(n+1) < r < r*}$$

$$r^* = \frac{n - \sqrt{(1-M_N^2)\left(1-\frac{\rho_u}{\rho_N}\right)}}{n+1} \qquad \frac{1}{2\overline{M}_u^2-1} < \frac{1}{\overline{M}_u^2} < \frac{\overline{M}_u^2 - (\overline{M}_u^2-1)^2\sqrt{2\overline{M}_u^{-2}[2\gamma\overline{M}_u^2-(\gamma-1)]^{-1}}}{2\overline{M}_u^2-1}$$



Clavin Williams 2012

Shock waves in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws $1/t^{3/2}$

OK with experiments

Freeman 1957 Lapworth 1959

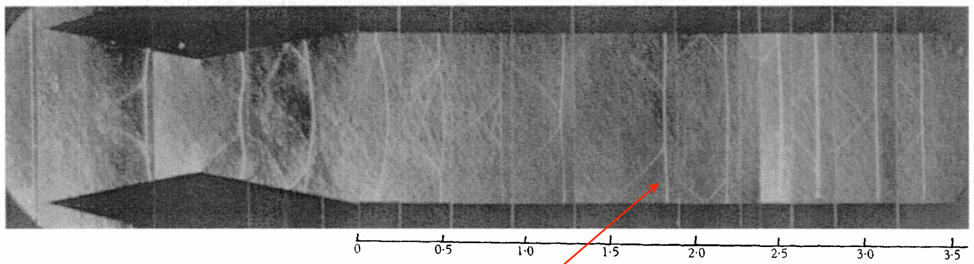


FIGURE 6 (plate 1). Photographs of shock wave at $M_s = 1{\cdot}41$.

Formation of Mach stems
(see next lecture)

Stability of reacting shocks

Clavin Williams 2009 2013

Reacting shocks = detonations considered as an hydrodynamic discontinuity

thickness = 0: no modification of the inner structure

$$r \equiv -\frac{\rho_u \mathcal{D}^2}{\mathrm{d}p_b/\mathrm{d}\rho_b^{-1}} = \overline{M}_u^2 \frac{1+\overline{\mathcal{V}}_b}{1-\overline{M}_u^2 \overline{\mathcal{V}}_b} \qquad \overline{\mathcal{V}}_b = \left[\overline{M}_u^{-2} - (1+\chi)\right]/2 \qquad \chi \equiv \sqrt{\left(1-\overline{M}_u^{-2}\right)^2 - 4\mathcal{Q}\overline{M}_u^{-2}} \qquad r = \frac{(1-\chi)+\frac{1}{\overline{M}_u^2}}{(1+\chi)+\frac{1}{\overline{M}_u^2}}$$

$0 \leqslant \chi < 1$

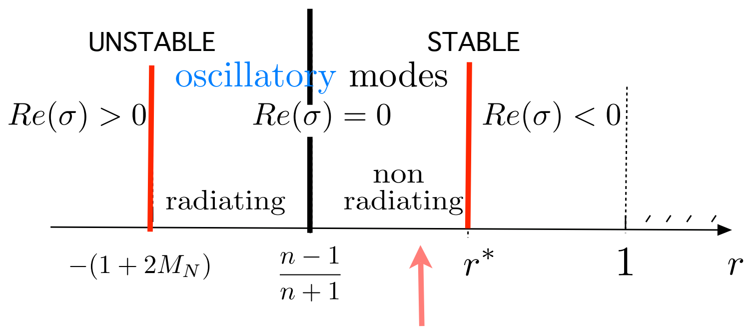
CJ wave $\nearrow$ overdriven regime

$$\chi = \sqrt{\left(1-\overline{M}_u^{-2}\right)^2 - \left(1-\overline{M}_{u_{CJ}}^{-2}\right)^2 \overline{M}_{u_{CJ}}^2 \overline{M}_u^{-2}}$$

$$n \equiv \left(\frac{\overline{\rho}_b}{\rho_u}\right) \frac{\overline{M}_b^2}{1-\overline{M}_b^2} == \frac{1}{\overline{M}_u^{-2} - 1 - 2\overline{\mathcal{V}}_b} = \frac{1}{\chi}$$

$$r = \frac{(1-\chi)+\frac{1}{\overline{M}_u^2}}{(1+\chi)+\frac{1}{\overline{M}_u^2}} = \left(\frac{n-1}{n+1}\right) \left[\frac{1+\frac{1}{\overline{M}_u^2(1-\chi)}}{1+\frac{1}{\overline{M}_u^2(1+\chi)}}\right]$$

$$\boxed{\left(\frac{n-1}{n+1}\right) \leqslant r}$$



Clavin Williams 2009 $\neq$ Majda Rosales 1983

Overdriven reacting shocks in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws

For the CJ marginal regime the acoustic waves in the burned gas propagate in the direction parallel to the unperturbed planar solution