TURBULENCE MODELING (ii)
Variable density flows

AIM: To discuss some specific features in turbulence modeling of variable density flows and present some closure schemes of single-point, first and second-order statistical moments equations.

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12. CONCLUDING REMARKS
1. MEAN MOTION EQUATIONS

Mass-weighted (Favre’s averages) are used throughout the present chapter.

Mean balance equations (binary mixture of perfect gases)

\[
\begin{align*}
\text{Continuity} & : \quad \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \bar{U}_k)}{\partial x_k} = 0 \\
\text{Momentum} & : \quad \frac{\partial (\bar{\rho} \bar{U}_i)}{\partial t} + \frac{\partial (\bar{\rho} \bar{U}_i \bar{U}_j)}{\partial x_j} = \bar{\rho} \ddot{U}_i - \frac{\partial \bar{P}}{\partial x_i} - \frac{\partial (\bar{\rho} \bar{u}_i \bar{u}_j)}{\partial x_j} + \frac{\partial \bar{\tau}_{ij}}{\partial x_j} \\
\text{Internal Energy} & : \quad \frac{\partial (\bar{\rho} \bar{e})}{\partial t} + \frac{\partial (\bar{\rho} \bar{e} \bar{U}_j)}{\partial x_j} = \bar{\rho} \ddot{U}_i - \bar{P} \frac{\partial \bar{u}_i}{\partial x_i} + \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_i} - \frac{\partial (\bar{P} \bar{e} \bar{u}_j)}{\partial x_j} \\
\text{Mass-fraction} & : \quad \frac{\partial (\bar{\rho} c)}{\partial t} + \frac{\partial (\bar{\rho} c \bar{U}_j)}{\partial x_j} = - \frac{\partial (\bar{\rho} \bar{e} \bar{u}_j)}{\partial x_j} + \frac{\partial \bar{q}_{mj}}{\partial x_j}
\end{align*}
\]

\( \bar{\tau}_{ij} \) is the viscous stress tensor, \( q_j \ (j = 1, 2, 3) \) the molecular heat flux, \( q_{mj} \ (j = 1, 2, 3) \) the molecular mass flux.

The pressure-dilatation correlation

\[
\bar{P} \frac{\partial \bar{u}_i}{\partial x_i} = \bar{P} \frac{\partial \bar{u}_i}{\partial x_i} + \left( p \frac{\partial \bar{u}_i}{\partial x_i} \right)
\]

\( \Pi_d \equiv p' \theta' = p' \frac{\partial \bar{u}_i}{\partial x_i} \)

Note that \( p' \frac{\partial \bar{u}_i}{\partial x_i} \equiv p' \frac{\partial \bar{u}_i}{\partial x_i} \)

The dilatation-dissipation:

\[
\bar{\tau}_{ij} \frac{\partial \bar{u}_i}{\partial x_i} = \bar{\tau}_{ij} \frac{\partial \bar{u}_i}{\partial x_i} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_i}
\]

Discarding viscosity fluctuations:

\[
\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_i} \equiv (\bar{P} \bar{e}) = \bar{\rho} \bar{e}_s + \bar{\rho} \bar{e}_d + \bar{\rho} \bar{e}_nh
\]

Solenoidal dissipation

\[
\bar{\epsilon}_s = 2 \frac{\bar{\mu}}{\bar{P}} \omega' \omega_j
\]

Dilatational dissipation

\[
\bar{\epsilon}_d = 4 \frac{\bar{\mu}}{3} \bar{\rho} \theta'^2
\]

Non-homogeneous dissipation

\[
\tau_{nh} = 2 \frac{\bar{\mu}}{\bar{P}} \left( \frac{\partial^2 \bar{u}_i \bar{u}_j}{\partial x_i \partial x_j} - 2 \frac{\partial (\bar{\theta} \bar{u}_j)}{\partial x_j} \right)
\]

Here, \( \omega' = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right) \)
2. THE MEAN-FLOW CLOSURE PROBLEM

- **Formally**, three types of terms in the open set of mean-motion equations are reflecting *explicit* effects of density variations:
  
  - Analogous, by Favre's extension, to the isovolume situation (blue) -> to be modeled
  
  - Specific to non-isovolume mean flow (green) -> exact
  
  - Specific to non-zero turbulent mass fluxes (red) [pressure-dilatation correlation, dilatational or compressible dissipation] -> to be modeled.

3. THE EDDY-VISCOSITY REPRESENTATION

- Direct extensions from isovolume:
  
  Jones & Launder 1972 (*M*-238)
  Jones 1979 (*M*-237)
  Ha Minh *et al.* 1981 (*M*-195)

  \[
  - \bar{\rho} u_i^i u_j^j + 2 \bar{\rho} k \delta_{ij} = 2 \mu (S_{ij} - \frac{1}{3} \frac{\partial \bar{U}_i}{\partial x_l} \delta_{ij})
  \]

- Compressible adaptations:
  
  Wilcox & Rubesin 1980 (*M*-485)
  (k-\(w^2\)) model - \(w = \varepsilon/k\) -
  Dussauge & Quine 1988 (*M*-141)
  (k-\(\varepsilon\)) model
  Taulbee & VanOsdl 1991
  (*M*-454) (k - \(\varepsilon - \rho \frac{\partial u}{\partial x} - u_i u_i^2\) model

  \[
  - \bar{\rho} u_i^i u_j^j + 2 \bar{\rho} k \delta_{ij} - 2 \mu (S_{ij} - \frac{1}{3} \frac{\partial \bar{U}_i}{\partial x_l} \delta_{ij})
  \]

  \[
  - \bar{\rho} u_i^i u_j^j + \frac{2}{3} \bar{\rho} \bar{k} \delta_{ij} + \frac{1}{3} \rho \bar{u}_i^i \bar{u}_j^j \delta_{ij}
  \]

  ( Taulbee & VanOsdl)

  "Incompressible" expressions of the turbulence eddy viscosity are used to close the previous schemes.
4. COMPRESSIBILITY EFFECTS ON THE EDDY-VISCOSITY SCHEME

Consider eq.(1)

\[
\nu_t = C_\mu \frac{k}{c} \equiv C_\mu \sqrt{\frac{k}{\tau}} \times \ell \quad \text{with} \quad \ell = \frac{k}{\tau}
\]

The 'incompressible' value \( C_\mu = 0.09 \) is a function of the turbulence Mach number \( M_t = \sqrt{\frac{k}{c}} \), Nichols 1990 (M-346)

In compressible flows, the use of a single length-scale is questionable:

- Zeman & Coleman 1991 (M-499) in compressed turbulence,
- Jamme 1998 (M-231) in shock-turbulence interaction,
- Freund et al. 2000 (M-163) in a compressible mixing layer.

Modified Boussinesq's scheme: Yoshizawa et al. 1997 (M-492)

\[
\frac{\bar{u}_i \bar{u}_j}{3} - \frac{2}{3} \bar{k} \delta_{ij} = -2 \nu_c (\bar{S}_{ij} - \frac{1}{3} S_{mm} \delta_{ij}) + \text{non-linear correction} + \frac{\nu_t}{[1 + A(I_\rho/M_t)]^{3/2}} \nu_t \quad \text{with} \quad \nu_c = \frac{1}{[1 + A(I_\rho/M_t)]^{3/2}} \nu_t \quad \text{from eq.(1),} \quad M_t = \sqrt{\frac{k}{c}}, \quad I_\rho = \sqrt{\rho^2/\bar{\rho}}, \quad A \approx 5 - 10
\]

Non linear constitutive relation: Wilcox & Rubesin 1980 (M-485)

\[
- \bar{\rho} \bar{u}_i \bar{u}_j'' + \frac{2}{3} \bar{\rho} \bar{k} \delta_{ij} = 2 \mu_c (\bar{S}_{ij} - \frac{1}{3} \partial U_l \partial x_l \delta_{ij}) + \frac{8}{9} \frac{\bar{S}_{im} \bar{R}_{mj} + \bar{S}_{jm} \bar{R}_{mi}}{\beta^* \omega^2 + 2 \bar{S}_{mm} \bar{S}_{nm}}
\]

with: \( \bar{R}_{ij} = \frac{1}{2} \left( \frac{\partial \bar{U}_k}{\partial x_j} - \frac{\partial \bar{U}_j}{\partial x_k} \right), \omega = \frac{\bar{c}}{k}, \) and \( \beta^* = 0.09 \) a model coefficient.
5. MODELING TURBULENT HEAT/MASS TRANSPORT

- Gradient diffusion scheme

\[- \tau f' u''_i = \tau \frac{\partial F}{\partial x_j} \]

with \( I_t = \nu_t \times \sigma_t \), where \( \sigma_t \) is a turbulence Prandtl/Schmidt number:

- \( \sigma_t \approx 0.8 \) to 0.9 : Flat plate boundary layer
- \( \sigma_t \approx 0.7 \) : jet
- \( \sigma_t \approx 0.5 \) : wake

6. MODELING d.f.c.

\( \rightarrow \) Turbulent mass flux \( \overline{w_i} = - \rho' u'_i / \rho \) and scalar d.f.c. \( \overline{f'} \equiv - \rho' f' / \rho \)

have no equivalent in constant density flows

6.1 EXACT EXPRESSIONS

- Isothermal and isobaric mass mixing (a)

From \( \rho = \rho_0 C + b \ (a, b \ \text{constants}) \):
\[
\rho_0' + \rho_0' = a \left( \frac{a C}{\rho_0' + \rho_0' + \rho_0'} \right) + b
\]

Hence
\[
\rho' f' = \frac{a}{1 - aC} \rho_0' f'
\]

- Isobaric temperature mixing (b)

From \( P\rho = RT \ (R \ \text{constant}) \):
\[
\rho' = - \left( \frac{1}{T} \rho_0' - \rho_0' \right)
\]

Hence
\[
\rho' f' = - \frac{1}{T} \rho_0' f'
\]
Application to jet and wake flows

\[ \frac{\rho' \gamma'}{\rho' \gamma'^2} = \frac{\rho u_i' \gamma'}{\rho u_i' \gamma'^2} = \frac{a}{1 - aC} \]  
\[ \frac{\rho' \theta'}{\rho' \theta'^2} = \frac{\rho u_i' \theta'}{\rho u_i' \theta'^2} = -\frac{1}{T} \]

Temperature-density fluctuations are always negatively correlated;

Mass-fraction-density fluctuations are negatively correlated in 'light' jets and positively correlated in 'heavy' jets.

6.2 LINEARIZED APPROXIMATIONS

From the equation of state:

- Perfect gas, homogeneous composition
- Polytropic evolution
- Isothermal and isobaric inhomogeneous mixing

\[ \frac{1}{\rho} = R \frac{T}{P} \]

\[ P = C^t \times \rho^n \]

\[ \frac{1}{\rho} = \frac{C}{\rho_1} + \frac{1 - C}{\rho_2} \]

\( R, \) constant

\( n = 0 \) isobaric
\( n = 1 \) isothermal
\( n = C_p/C_v \) isentropic

\( \rho_1, \rho_2 \) constants
assuming $\rho'/\bar{\rho} \ll 1$, $\theta'/\bar{\theta} \ll 1$, $\gamma'/\bar{\gamma} \ll 1$ ($\theta' = T - \bar{T}$, $\gamma' = C - \bar{C}$)

\[
\begin{align*}
\frac{\rho'}{\rho} &\approx \frac{\rho'}{\bar{\rho}} - \frac{\theta'}{\bar{T}} \\
\frac{n'}{n} &\approx \frac{n'}{\bar{n}} \\
\frac{\rho'}{p} &\approx \frac{\rho_1 - \rho_2}{\rho_1 \rho_2} \gamma'
\end{align*}
\]

Hence, the turbulent mass fluxes, for instance:

\[
\begin{align*}
\frac{\rho'u_i}{\bar{p}} &\approx \frac{\rho'u_i}{\bar{p}} - \frac{\theta'u_i}{\bar{T}} \\
\frac{n'u_i}{n} &\approx \frac{n'u_i}{\bar{n}} \\
\frac{\rho'u_i}{p} &\approx \frac{\rho_1 - \rho_2}{\rho_1 \rho_2} \gamma'
\end{align*}
\]

In particular, for a non-isothermal polytropic evolution of a perfect gas ($n \neq 1$)

\[
\frac{\rho'u_i}{\bar{p}} \approx \frac{1}{n-1} \frac{\theta'u_i}{\bar{T}}
\]

In addition: Examples

<table>
<thead>
<tr>
<th>Author / Ref.</th>
<th>Scheme</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rubesin 1976 (M-397)</td>
<td>$\frac{\rho'u_i}{\bar{p}} \approx - \frac{\bar{U}}{(n-1)C_pT} \frac{u_i^2}{\bar{U}}$</td>
<td>Constant Total-Temperature Turbulent mass flux systematically negative</td>
</tr>
<tr>
<td>Galmes et al. 1983 (M-174)</td>
<td>$\frac{\rho'u_i}{\bar{p}} = R_{pa}(\gamma - 1)M^2 \frac{u_i^{n_2}}{\bar{U}}$</td>
<td>Strong Reynolds analogy Supersonic B.L. with non-zero pressure gradient</td>
</tr>
<tr>
<td>Vandromme &amp; Ha Minh 1985 (M-332)</td>
<td>$\bar{u}_j = \frac{\bar{U}_j}{(n-1)C_pT}$</td>
<td>Various compressible flows, including shock waves</td>
</tr>
<tr>
<td>Dussauge &amp; Quine 1988 (M-141)</td>
<td>$\frac{\rho'u_i}{\bar{p}} = C(i)(\gamma - 1)M^2 \frac{u_i^{n_2}}{\bar{U}}$</td>
<td>2-D mixing -layer The model constant $C(i)$ is adjusted to the flux component: $C(1) = 0.8$ and $C(2) = 1.5$</td>
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6.3 GRADIENT DIFFUSION SCHEMES

- General expression: \[- \rho' u_i' = D_i \frac{\partial \rho}{\partial x_i}\]

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<thead>
<tr>
<th>Author/Ref.</th>
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<th>Remarks</th>
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<tbody>
<tr>
<td>✓ Milinazzo &amp; Saffman 1976 (M-331)</td>
<td>(- \rho' u_i' = \frac{1}{\sigma_p} \frac{\partial \rho}{\partial y})</td>
<td>2-D inhomogeneous, low-speed mixing layer</td>
</tr>
<tr>
<td>✓ Taulbee &amp; VanOsldol 1991 (M-454)</td>
<td>(\rho' u_i' = - \frac{\nu_t}{\sigma_p} \frac{\partial \rho}{\partial x_i})</td>
<td>Compressible flows</td>
</tr>
<tr>
<td>✓ Sarkar &amp; Lakshmanan 1991 (M-413)</td>
<td>(\theta' u_i' = - \frac{\nu_t}{\sigma_T} \frac{\partial \theta}{\partial x_i})</td>
<td>Same Prandtl numbers for density and temperature fluxes (0.7).</td>
</tr>
</tbody>
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6.4 D.f.c. TRANSPORT EQUATION

<table>
<thead>
<tr>
<th>Author/Ref.</th>
<th>Model</th>
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<tbody>
<tr>
<td>✓ Jones 1979 (M-237):</td>
<td>(\rho(\frac{\partial \bar{u}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{u}_i}{\partial x_j}) = ) Production + Diffusion - Destruction</td>
</tr>
<tr>
<td>✓ Taulbee &amp; VanOsldol 1991 (M-454)</td>
<td>Production = (- \bar{\rho} \bar{u}_j \frac{\partial \bar{U}_i}{\partial x_j} + \frac{1}{\rho} (\rho \bar{u}_j \bar{u}_j - \rho' u_j' u_j') \frac{\partial \rho}{\partial \bar{U}_j})</td>
</tr>
<tr>
<td></td>
<td>Diffusion = (\frac{\partial}{\partial x_j} [\mu_i \frac{\partial \bar{u}_j}{\partial x_j} + \frac{\partial \bar{u}<em>j}{\partial x_i} - \frac{2}{3} \frac{\partial \bar{u}<em>m}{\partial x_m} \delta</em>{ij}] + \frac{1}{3} \rho' u_m' \delta</em>{ij})</td>
</tr>
<tr>
<td></td>
<td>Destruction = (C_{u_2} \bar{\rho} \frac{\varepsilon}{k} u_i)</td>
</tr>
</tbody>
</table>
7. PRESSURE-DILATATION CORRELATION

\[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}\bar{\theta}'} \text{ from an algebraic relation} \quad \Rightarrow \text{Zero-equation model} \]

\[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}\bar{\theta}'} \text{ from an additional transport equation} \quad \Rightarrow \text{One-equation model} \]

### 7.1. ZERO-EQUATION MODEL

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<tr>
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<tbody>
<tr>
<td>√ Viegas &amp; Horstman 1978 (M-476)</td>
<td>[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}\bar{\theta}'} = \xi \frac{k}{\gamma} M^2 \frac{\partial \bar{U}_i}{\partial x_i} ]</td>
<td>Constant Total-Temperature assumption, linearized for temperature fluctuations. ( M ) is the local Mach number, ( \xi = 0.73 )</td>
</tr>
<tr>
<td>√ Horstman 1987 (M-218)</td>
<td>[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}\bar{\theta}'} = C \frac{n}{\gamma} \left( \frac{\gamma - 1}{n - 1} \right)^2 \frac{k}{\gamma} M^2 \frac{\partial \bar{U}_i}{\partial x_i} ]</td>
<td>( M ) is the local Mach number, ( n = 1.2 ), ( C = 0.12 )</td>
</tr>
<tr>
<td>√ Sarkar 1992 (M-408)</td>
<td>[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}\bar{\theta}'} = \alpha_2 M_t \frac{\partial \bar{U}<em>i}{\partial x_j} \alpha</em>{ij} k + \frac{8}{3} \alpha_4 M_t^2 \frac{\partial \bar{U}_j}{\partial x_j} k + \alpha_3 M_t^2 \frac{\partial \bar{U}_j}{\partial x_j} k ]</td>
<td>( M_t = \sqrt{\frac{2k}{\epsilon}} ) is the turbulence Mach number, ( \epsilon ) is the solenoidal dissipation, ( a_{ij} = u_i' u_j' - (2k \delta_{ij}/3) ),</td>
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### 7.2. ONE-EQUATION MODEL

<table>
<thead>
<tr>
<th>Author/Ref.</th>
<th>Function</th>
<th>Model / Conditions</th>
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<tbody>
<tr>
<td>√ Aupoix et al. 1990 (M-24)</td>
<td>( \bar{k} )</td>
<td>[ \frac{D \bar{k}}{Dt} = -C_1 \frac{1}{\tau_{\alpha}} M_t^2 \frac{D \bar{k}}{Dt} - C_2 \frac{1}{\tau_{\alpha}} \bar{p}' \bar{\theta}' ] Homogeneous sheared compressible turbulence, ( M_t = \sqrt{\frac{2k}{c}} ), ( \tau_{\alpha} = \frac{\bar{k}^{3/2}}{\epsilon c} ), ( C_1 = 0.25 ), ( C_2 = 0.2 )</td>
</tr>
<tr>
<td>√ Zeman 1991 (M-497)</td>
<td>( \bar{p}/\bar{k} )</td>
<td>[ \frac{\bar{p}'\bar{\theta}'}{\bar{p}^{2}} = -\frac{1}{2\bar{p} c^2} \frac{D \bar{p}^{2}}{Dt} ] Free-decaying and shear driven homogeneous turbulence</td>
</tr>
<tr>
<td>√ Hamba 1999 (M-203)</td>
<td>( \bar{k}, \bar{p}/\bar{k} )</td>
<td>[ \frac{\bar{p}'\bar{\theta}'}{\bar{k}, \bar{p}/\bar{k}} = -\left( 1 - C_p T_{\alpha} \chi_p \right) \times \left[ C_{pd1} M_t^2 \frac{D \bar{p}}{Dt} + C_{pd2} \gamma M_t^2 \frac{\partial \bar{U}_i}{\partial x_i} \right] ] ( \chi_p = \bar{p}/(2\bar{p}^2\bar{c}^2) )</td>
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</table>
In addition:

1) According to Ristorcelli 1997 (M-394) the pressure-dilatation is found to be a non-equilibrium phenomenon. Restricted to low $M_t^2$ situations, the pressure-dilatation correlation is important in flow fields where the turbulence kinetic energy is far from an equilibrium state "Production = Dissipation".

2) The pressure-dilatation correlation plays an important role in the pressure variance equation, so that the modeling of this term is to be addressed when deriving a closure expression to this equation.

In compressible homogeneous shear flow, for instance, the DNS results of Hamba 1999 (M-203) show that the pressure-dilatation correlation is the dominant term in the pressure variance equation.

### 8. DILATATION-DISSIPATION

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Zeman 1990</td>
<td>$\overline{\tau}_d = c_d F(M_t, K) \overline{\alpha}_s$</td>
<td>$M_t = \sqrt{2k/\overline{\epsilon}}, \overline{\epsilon} = \sqrt{\gamma R T}$, $K = u''^4/(u''^2)^2$</td>
</tr>
<tr>
<td>Sarkar et al. 1991</td>
<td>$\overline{\tau}_d = \alpha_1 M_t^2 \overline{\tau}_s$</td>
<td>$M_t = \sqrt{2k/\overline{\epsilon}}, \alpha_1 = 1$</td>
</tr>
</tbody>
</table>

- Solenoidal dissipation:

  $\overline{\tau}_s \propto \frac{k^{3/2}}{l}$

In addition:

Within the frame of the analysis developed by Ristorcelli 1997 (M-394), the ratio $\overline{\tau}_d/\overline{\tau}_s$ is found to be a function of the turbulence Reynolds number $R_t = 4 \overline{\epsilon}^2/9 \nu \overline{\epsilon}$. Moreover, a representation for the effects of the compressible (or dilatational) dissipation $\overline{\tau}_d$ consists in a sum of slow ($\overline{\tau}_d^s$) and rapid ($\overline{\tau}_d^r$) portions scaling as

$$\frac{\overline{\tau}_d^s}{\overline{\tau}_s} \propto \frac{M_t^4}{R_t} \quad \text{and} \quad \frac{\overline{\tau}_d^r}{\overline{\tau}_s} \propto \frac{M_t^2 M_S^2}{R_t}$$

for high-$R_t$ and low- $M_t^2$ non-equilibrium flows respectively.

Here, $M_t$ and $M_S$ are the turbulence and strain (gradient) Mach numbers respectively.

An important feature of the previous model is the dependence of the dilatation dissipation on the viscosity: for fixed $M_t$, $\overline{\tau}_d$ vanishes at a sufficiently high turbulence Reynolds number.
9. **FIRST ORDER MODELING**

9.1. **THE T.K.E. EQUATION**

\[ \tilde{k} = \frac{1}{2} \tilde{u}^\alpha \tilde{u}^\alpha \equiv \frac{1}{2} \frac{\rho \tilde{u}_i \tilde{u}_j}{\tilde{p}} \]

**Modeled transport equation**

\[ \rho \left( \frac{\partial \tilde{k}}{\partial t} + \nabla \cdot \tilde{U}_j \right) = - \rho \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{U}_i}{\partial x_j} - \frac{\partial (\tau_{ij} \tilde{u}_j)}{\partial x_i} + \frac{1}{2} \left[ \frac{\partial (\rho \tilde{u}_i \tilde{u}_j)}{\partial x_j} + \frac{\partial (p' \tilde{u}_j)}{\partial x_j} - \tilde{u}_i \frac{\partial \tilde{p}}{\partial x_i} + p' \frac{\partial \tilde{u}_i}{\partial x_i} \right] \]

**Modeled eq.:**

\[ \frac{D}{Dt} (\tilde{p} \tilde{k}) = P_{rod}^{(k)} + D_{ij}^{(k)} - \tilde{p} \tilde{\tau}_i + A_{\rho} \]

with \( \mu_t = C_\mu \tilde{p} \frac{\tilde{k}^2}{\tilde{\epsilon}_s} \)

**Production:**

\[ P_{rod}^{(k)} = - \rho \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{U}_i}{\partial x_j} = 2 \mu_t S_{ij} \frac{\partial \tilde{U}_i}{\partial x_j} - \frac{2}{3} \mu_t \left( \frac{\partial \tilde{U}_i}{\partial x_i} \right)^2 - \frac{2}{3} \tilde{p} \frac{\partial \tilde{U}_i}{\partial x_i} \]

**Diffusion:**

\[ D_{ij}^{(k)} = - \frac{\partial}{\partial x_j} \left[ \frac{1}{2} (\rho \tilde{u}_i \tilde{u}_j) + p' \tilde{u}_j \right] = \frac{\partial}{\partial x_j} \left( \frac{\mu_t}{\sigma_k} \tilde{\epsilon}_s \frac{\partial \tilde{k}}{\partial x_i} \right) \]

**Dissipation**

\[ \tilde{\epsilon}_s = 2 \frac{\mu_t}{\tilde{\rho}} \omega \tilde{\omega} \]

(solenoidal)

\[ \tilde{\epsilon}_s = 2 \frac{\mu_t}{\tilde{\rho}} \omega \tilde{\omega} \longrightarrow \text{Modeled transport equation} \]

**Specific contributions due to density variation:**

\[ A_{\rho} = - \tilde{u}_i \frac{\partial \tilde{p}}{\partial x_i} + p' \frac{\partial \tilde{u}_i}{\partial x_i} + \tilde{\epsilon} \frac{\partial \tilde{\tau}_{ij}}{\partial x_j} - \tilde{p} \tilde{\epsilon}_d - \tilde{p} \tilde{\epsilon}_{nh} \]

9.2. **THE DISSIPATION EQUATION**

\[ (\tilde{\epsilon}_s \equiv \tilde{\epsilon}) \]

**Modeled form:**

\[ \frac{\partial (\tilde{p} \tilde{\epsilon}_s)}{\partial t} + \frac{\partial (\tilde{p} \tilde{\epsilon}_s \tilde{U}_j)}{\partial x_j} = - C_1 \frac{\tilde{\epsilon}_s}{\tilde{k}} \frac{\rho}{\tilde{p}} \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial (\tilde{\epsilon}_s \frac{\partial \tilde{U}_i}{\partial x_j})}{\partial x_j} \frac{\tilde{p}}{\sigma_e} \frac{\partial \tilde{p}}{\partial x_j} - C_2 \tilde{\epsilon}_s \frac{\tilde{p}^2}{\tilde{k}} \frac{\partial \tilde{\epsilon}_s}{\partial x_j} + \frac{\tilde{p}}{\tilde{k}} \frac{\partial \tilde{\epsilon}_s}{\partial x_j} + C_3 \tilde{\epsilon}_s \frac{\tilde{p}^2}{\tilde{k}} \frac{\partial \tilde{\epsilon}_s}{\partial x_j} + C_4 \tilde{\epsilon}_s \frac{\tilde{p}^2}{\tilde{k}} \frac{\partial \tilde{\epsilon}_s}{\partial x_j} + C_5 \frac{\tilde{p}^2}{\tilde{k}} \frac{\partial \tilde{\epsilon}_s}{\partial x_j} \]

\( (d) \) pressure-dilatation counter-part from the T.K.E. equation,

\( (e) \) mean pressure counter-part from the T.K.E. equation,

\( (f) \) specific contribution accounting for the dependence of the turbulence length scale on passing through a shock wave.

9.3. **THREE-EQUATION MODELS**

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10. SECOND-ORDER MODELING

10.1. THE OPEN REYNOLDS STRESS TRANSPORT EQUATION

\[
\frac{\partial (\bar{\rho} \bar{u}_i \bar{u}_j)}{\partial t} + \frac{\partial (\bar{\rho} \bar{u}_i \bar{u}_j \bar{U}_k)}{\partial x_k} = \bar{\rho} P_{ij} - \frac{\partial (\bar{T}_{ijk})}{\partial x_k} + \bar{\rho} T_{ij} - \bar{\rho} \bar{\varepsilon}_{ij} - \frac{2}{3} \bar{\rho} \bar{\tau}_s \delta_{ij} + \Sigma_{ij} + \frac{2}{3} \rho' \theta' - \bar{\rho}(\bar{\tau}_d + \bar{\tau}_{nh}) \delta_{ij}
\]

Mean-flow coupling ("Turbulent production")
\[
P_{ij} = -\left( \frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j} \right)
\]

Transport by fluctuations ("Turbulent diffusion")
\[
T_{ijk} = \frac{\partial u_i'' u_j'' u_k''}{\partial x_k} + \frac{\partial u_i'}{\partial x_j} \delta_{jk} + \frac{\partial u_j'}{\partial x_i} \delta_{ik} - \left( \tau_{jk} u_i'' + \tau_{ik} u_j'' \right)
\]

Deviatoric part of the pressure-strain correlation
\[
\Pi_{ij} = \frac{1}{\bar{\rho}} \left[ \frac{p'}{\partial x_i} + \frac{\partial u_j''}{\partial x_i} \right] - \frac{2}{3} \frac{\rho' \theta'}{\bar{\rho}} \delta_{ij}
\]

Deviatoric part of the dissipation tensor
\[
\bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij} - \frac{2}{3} \bar{\rho} \bar{\varepsilon}_{ij} \quad \text{where} \quad \bar{\rho} \bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ij} = \left( \tau_{jk} \frac{\partial u_i''}{\partial x_k} + \tau_{ik} \frac{\partial u_j''}{\partial x_k} \right)
\]

Mass Flux coupling
\[
\Sigma_{ij} = u_i'' \left( \frac{\partial \tau_{jk}}{\partial x_k} - \frac{\partial \bar{P}}{\partial x_j} \right) + u_j'' \left( \frac{\partial \tau_{ik}}{\partial x_k} - \frac{\partial \bar{P}}{\partial x_i} \right)
\]

Pressure-dilatation correlation
\[
\frac{2}{3} \rho' \theta' \delta_{ij} \quad \text{where} \quad \theta' = \frac{\partial u_i'}{\partial x_i}
\]

Dilatation and non-homogeneous dissipation
\[
\bar{\varepsilon}_{ij} = \frac{4}{3} \frac{\bar{\rho}}{\rho} \bar{\theta}'^2 \bar{\varepsilon}_{ij} = \frac{1}{\bar{\rho}} \left( \frac{\partial^2 u_i'' u_j''}{\partial x_i \partial x_j} - \frac{2}{3} \frac{\rho' \theta'}{\bar{\rho}} \frac{\partial u_i''}{\partial x_i} \right)
\]

10.2 MODELING THE PRESSURE-STRAIN CORRELATION

\[
\Pi_{ij} = \frac{1}{\bar{\rho}} \left[ \frac{p'}{\partial x_i} + \frac{\partial u_j''}{\partial x_i} \right] - \frac{2}{3} \frac{\rho' \theta'}{\bar{\rho}} \delta_{ij}
\]

\[\leftrightarrow \quad \Pi_{ii} \equiv 0 \quad \text{Redistribution between normal components at a constant T.K.E. level}\]

\[
\Delta p' = -2 \frac{\partial^2 (\rho u_i' \bar{U}_j)}{\partial x_i \partial x_j} - \frac{\partial^2 (\rho u_i' u_j' - \rho u_i' u_j')}{\partial x_i \partial x_j} + \frac{\partial^2 \tau'_{ij}}{\partial x_i \partial x_j} + g_c \frac{\partial p'}{\partial x_i} - 2 \frac{\partial^2 \left( \rho' u_i' - \rho u_i' \right) \bar{U}_j}{\partial x_i \partial x_j} + \frac{\partial^2 \rho'}{\partial x_i \partial x_j} - \frac{\partial^2 (\rho' \bar{U}_i \bar{U}_j)}{\partial x_i \partial x_j}
\]
Model: General Form
\[ \Pi_{ij} = \Pi_{ij}^{\text{rapid}} + \Pi_{ij}^{\text{low}} + \Pi_{ij}^{\text{p}} \]

- **Buoyant flows**  
  Launder, 1984 (M-274), Craft, 1991 (M-109)

Isotropization of production
\[ \Pi_{ij}^{\text{p}} = - C_{3}(G_{ij} - \frac{1}{3}G_{mm}\delta_{ij}) \]

- \[ -> \] Buoyant generation of Reynolds stresses:
\[ G_{ij} = -\frac{1}{T}(\bar{\theta}u'_{i}g_{j} + \bar{\theta}u'_{j}g_{i}) \equiv \frac{1}{\rho}(\rho'u'_{i}g_{j} + \rho'u'_{j}g_{i}) \]

- **Low-speed ideal binary mixing**
  
  ✓ Helium-nitrogen mixing layer: Shih, Lumley & Janicka, 1987 (M-427),
  
  \[ -> \] No significant density effects in the incompressible model

✓ Vertically expanding free jets: Chassaing, 1979 (M-80), Bailly Champion & Garréton, 1996 (M-27), Vallet, 1997 (M-465)

Mean pressure contribution
\[ \Pi_{ij} = b_{4}\frac{1}{\rho}(u'_{i}\frac{\partial \bar{P}}{\partial x_{j}} + u'_{j}\frac{\partial \bar{P}}{\partial x_{i}} - \frac{2}{3}u'_{k}\frac{\partial \bar{P}}{\partial x_{k}} \delta_{ij}) \]

where \( b_{4} = 0.3 \) to 0.75 is a model coefficient.

\[ -> \] Mean pressure accounts for gravity generation, since (with \( g_{i} = g\delta_{3i} \)):
\[ -\frac{u'_{j}}{\rho} \frac{\partial \bar{P}}{\partial x_{i}} \delta_{3i} \equiv \frac{\rho' u'_{j}}{\rho} \frac{\partial \bar{P}}{\partial x_{i}} \delta_{3i} = \rho' u'_{j} g_{i} \delta_{3i} \]
11. EXAMPLES

11.1 BUOYANT MIXING LAYER

- Spatially evolving, horizontal, saline mixing layer. The buoyant flow is stably stratified.

Craft, Ince & Launder, 1996 (M-112)
11.2 BUOYANT PLUME-JET

- Plume-jet flow: a hot jet discharging vertically downwards into a cold water flow moving upwards at less than 2% of the jet speed.

Craft, Ince & Launder, 1996 (M-112)


11.3 VARIABLE DENSITY FREE JETS

- Hot/cold or dense/thin free jets
  (Low-speed motion, no significant gravity effects)

Effect of the density ration at the exit:

\[ s_p = \frac{\rho_{jet}}{\rho_{exit}} \]

Self-preserving region:

\[ \frac{Q_m(x)}{Q_{m0}} = C_Q \frac{x}{s_p \sqrt{D_0}} \]
11.4 COMPRESSIBLE FREE MIXING LAYER

- Free-shear layer at increasing Mach number

**Convective Mach number**

$$M_c = \frac{U_1 - U_c}{a_1} \sim \frac{U_1 - U_2}{a_1 + a_2}$$

**Growth rate:**

$$\frac{d\delta}{dx} \approx \left(\frac{d\delta}{dx}\right)_{inc} \frac{U_1 - U_2}{2U_c} \times \phi(M_c)$$

Barre, Quine & Dussauge, 1994 (M-30)

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**Diagram:**

- Lioupsware & Laufer (1947)
- Bradshaw (1966)
- Wygnanski & Redler (1970)
- Isawa & Kubota (1976)
- Wagner (1979)
- Lau (1981)
- Chinesi et al. (1986)
- Patric, Samirny & Addy (1986)
- Samirny, Patris & Addy (1986)
- Samirny & Elliott (1990)
- Dutton et al. (1990)
- Elliott & Barhayan (1990)
- Hall (1991)
- Debischop (1993)
- Barre, Quine & Dussauge (1994)
11.5 SHOCK-INDUCED SEPARATION

- F2D-transonic flow over a bump
  (Délery test case C, Stanford Conf. 1981)

- "λ" shape shock:
- Standard 'k-ε' model:
- Modified 2nd order Launder-Shima model:

Leschziner, Batten & Loyau, 1997 (M-294)
Pressure distribution:
CONCLUDING REMARKS

General:

- Turbulence modeling of variable density flows addresses a wide range of situations, due to the variety of the origins of the density changes.

- As compared with the incompressible/isovolume situation, the closure of single-point statistical moments equations in variable density fluid turbulence is concerned with two types of terms:
  - Terms that have a formal "incompressible" counterpart,
  - Specific contributions which have no "equivalent" in the turbulent regime.

- When modeling first type of terms in variable density fluid turbulence, it is (generally implicitly) assumed that compressibility effects do not radically change the physics of the turbulent energy transfer, so that most of the incompressible closure schemes can be adapted to the equivalent compressible terms.

- In eddy-viscosity models, structural changes of the turbulence field associated with modifications in the Reynolds stress anisotropy tensor cannot be introduced explicitly into closure schemes, as it is the case with second-order modeling.

In high-speed flows, such modifications are not restricted to those usually encountered in the incompressible regime (e.g. shock-turbulence interaction)

- Assessing the influence of each type of contributions and appreciating the pertinence of incorporating all of them in a complete model are still open questions.
Low-speed, non-reactive fluid motions:

- Density variations arise from changes in temperature or/and composition which can produce high levels of density-intensity.
- Accordingly, one is faced with the modeling of turbulent mass flux and d.f.c. which have no equivalent in constant density flows.
- Free shear flows can be predicted by using direct extensions of closure schemes derived for incompressible and buoyant flows, provided a suitable closure for such d.f.c. is adopted.
- In this respect, simple gradient-type diffusion schemes can be totally irrelevant.

Channel flows and non-separated, high-speed boundary layers, up to free stream Mach numbers $M_\infty < 3$ to 5

- The effects of density and pressure fluctuations under adiabatic conditions at the wall are small.
- Hence, compressibility effects are mostly due to mean density and temperature variations, and mean velocity profiles can be recovered from that in incompressible turbulence, using the Van Driest transformation.
- To some extent, this explains why direct extensions of incompressible closure schemes to that of density-weighted moments for variable density flows ("first" generation), do not yield irrelevant predictions, as demonstrated by Ha Minh and coworkers.
- Even within the limit of Kovasznay's modes decomposition, density fluctuations are included in both acoustic ($\rho' \neq 0, \rho' \neq 0$ and $s'=\omega'=0$) and entropy ($s' \neq 0, \rho' \neq 0$ and $\omega'=p'=0$) modes.

With non-adiabatic walls, density fluctuations can exist with little compressible turbulence effects Huang, Coleman & Bradshaw, 1995 (M-221)

- Therefore, for boundary layers and channel flows, the density variance is not a suitable choice of an independent function to accurately account for compressibility effects in a $(k-\varepsilon)$ model,
for instance.

Compressible Free Shear Layer:

- In high-speed free shear flows, the first generation of models, based on direct extensions of incompressible closure schemes similarly to what adopted in predicting wall bounded flows, fail to predict the strong compressibility effects observed in such free flows.

- In a "second" generation, the closure addresses specific compressible contributions, viz the pressure-dilatation and the dilatational dissipation which were presumed to account for such effects.

- As shown by DNS of homogeneous isotropic turbulence, Erlebacher et al. 1990 (M-149), Sarkar et al. 1991 (M-412), Huang et. al. 1991 (M-221),... profound differences exist in the level of such dilatational terms in free and wall-bounded flows without shock;

- Since the publication by Zeman in 1990 (M-495), such terms and particularly the dilatational dissipation has been taken for a while as responsible for the stabilizing Mach number effect;

- In 1995, Sarkar (M-409) showed that the reduced growth rate of turbulence kinetic energy in homogeneous shear was primarily due to the reduced level of turbulence production, as a consequence of a change in the anisotropy of the Reynolds stress due to compressibility;

- As confirmed by Sarkar in 2002 "the contribution of dilatational terms to the turbulence kinetic energy balance remains small at Mach numbers at which substantial reduction in turbulence levels and thickness growth rate is observed."
... en route for Turbulence.