Propagation of elastic waves in a fluid-loaded anisotropic functionally graded waveguide: Application to ultrasound characterization

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Abstract

Non-destructive evaluation of heterogeneous materials is of major interest not only in industrial but also in biomedical fields. In this work, the studied structure is a three-layered one: a laterally heterogeneous anisotropic solid layer is sandwiched between two acoustic fluids. An original method is proposed to solve the wave equation in such a structure without using a multilayered model for the plate. This method is based on an analytical solution, the matricant, explicitly expressed under the Peano series expansion form. We validate this approach for the study of a fluid-loaded anisotropic and homogeneous plane waveguide with two different fluids on each side. Then, we give original results on the propagation of elastic waves in an asymmetrically fluid-loaded waveguide with laterally varying properties. This configuration notably corresponds to the axial transmission technique to the ultrasound characterization of cortical bone *in vivo*.

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I. INTRODUCTION

A lot of natural media have unidirectional varying elastic properties. The mantel crust, the oceans and cortical bone are some of these functionally graded media. Scientists focused on the advantages presented by this type of materials in terms of mechanical behavior and since the 80's, they developed industrial Functionally Graded Materials (FGM) particularly exploited in high-technology and biomedical applications. Consequently, the non-destructive evaluation of these materials is a key issue. Surface and guided waves play a major role in non-destructive testing and evaluation of complex structures. Several studies are dedicated to the leaky Lamb wave propagation in fluid-loaded plates (Chimenti and Nayfeh, 1990; Chimenti and Rokhlin, 1990; Deschamps and Poncelet, 2000). In all these studies, the media are homogeneous or multilayered. In this work, we introduce a general method to take into account the continuity of the property variation in an anisotropic waveguide. This method is based on the knowledge of an analytical solution of the wave equation, the matricant, explicitly expressed *via* the Peano series (Peano, 1888). The accuracy of the numerical evaluation of this solution and its validity domain are perfectly managed (Baron, 2005; Youssef and El-Arabawi, 2007). Because it deals with an analytical solution, all the wave propagation parameters are controlled. This represents an advantage with respect to numerical methods such as finite-element and finite-difference methods for which the problem treated is a global one, making difficult to analyze and interpret the experimental data which result from the interaction and coupling of numerous physical phenomena. To the best of our knowledge, this is the first method to evaluate the mechanical behavior of a fluid-loaded anisotropic waveguide with continuously varying properties without modelling the FGM plate by a multilayered plate. Consequently, in the context of real materials with continuously varying properties (such as bone, bamboo or manufactured FGM), this method allows to assess the solution to a more realistic model with a controlled accuracy and without an increase of the computation-time.

In this work, we first present the method and its setup with fluid-structure interaction; then we proceed to the validation of the method by comparing our results to the dispersion curves obtained from classical schemes on homogeneous waveguides (isotropic and anisotropic). Two advantages of the method are underlined: i) an asymmetric fluid-loading may be taken into account without modifying the scheme for the numerical solution;

ii) the influence of the property gradient on the ultrasonic response of the waveguide may be investigated *via* the frequency spectrum of the reflection coefficient *modulus*. Finally, we get onto the relevancy of this model applied to the ultrasound characterization of cortical bone by the axial transmission technique.

II. BACKGROUND

Contemporary work efforts over the last two decades illustrate some of the technology interest on guided waves to nondestructive evaluation. Namely, Rose (2002) gave a revue of ultrasonic guided wave inspection potential. A lot of papers deal with the interaction between guided waves and a solid plate immersed in a fluid or embedded between two different fluids. Guided modes in an infinite elastic isotropic plate in *vacuum* were first treated by Rayleigh (1885) then by Lamb (1917). The Lamb wave problem is reserved, strictly speaking, for wave propagation in a traction-free homogeneous isotropic plate. To deal with guided modes in a fluid-loaded plate, we use the term "leaky Lamb waves" as the energy is partly radiated in the fluids on both sides of the plate. For the basic Lamb problem –plate in *vacuum*–, the solutions of the dispersion equation are reals whereas in the case of a plate bounded by media on both sides, the dispersion equation has complex solutions. In 1961, Worlton (1961) gave an experimental confirmation of the theoretical work of Lamb, by obtaining experimentally the dispersion curves of aluminium and zirconium plates, asserting that water loading has little effect on the behavior of waves in plates. In 1976, Pitts *et al.* (1976) presented some numerical test results on the relationship between real part of the reflection coefficient poles and the phase velocity of leaky Lamb modes in a homogeneous isotropic brass plate in water. Folds and Loggins (1977) proposed analytical expressions of the reflection and transmission coefficients for plane waves at oblique incidence on a multilayered isotropic plate immersed in water based on Brekhovskikh's analysis (Brekhovskikh, 1980). They found good agreement with their theoretical results and experimental data. Few years later, Fiorito *et al.* (1979) developed a resonance formalism for the fluid-loaded elastic plate and gave some theoretical and numerical results for an isotropic steel plate immersed in the water. This formalism was generalized to the interactions of acoustic plane waves with an asymmetrically fluid-loaded elastic plate by Franklin *et al.* (2001). Nayfeh, Chimenti and Rokhlin produced a lot of works on wave propagation in anisotropic media and particularly in fiber composite plates immersed in a fluid (Chimenti and Nayfeh, 1986, 1990; Chimenti and Rokhlin, 1990; Nayfeh and Chimenti, 1988, 1989; Rokhlin and Wang, 2002). Based on their formalism, Deschamps and Poncelet (2000) placed the emphasis on the difference between what they called transient Lamb waves –solutions of the characteristic equation of the plate for complex frequency and real slowness (time attenuation)– and heterogeneous Lamb waves for which the slowness is complex and the frequency is real (spatial attenuation). These two ways of resolution of the dispersion equation have two different physical meanings –space or time attenuation– and so, different physical consequences developed in their paper. A critical point is the validity of the Cremer's coincidence hypothesis: the real couples (angular frequency ω and phase velocity v_{φ}), such that the reflection coefficient is minimum may be identified as velocity dispersion of plate waves. Experimentally checked in a lot of configurations, it appears to be not well satisfied in several cases (for instance, graphite-epoxy plates when the ratio between fluid and plate mass densities is not "small") (Chimenti and Nayfeh, 1986; Nayfeh and Chimenti, 1988). The results obtained by Deschamps and Poncelet (2000) on fluid-loaded plate show a good correlation between dispersion curves obtained in complex frequency and the *minima* of the reflection coefficient, which suggests that the Cremer's coincidence principle is still valid considering time attenuation. All these studies show evidence that the wave propagation in fluid-loaded elastic plate emerges as a very delicate problem which needs cautious treatment.

III. GENERAL FORMULATION OF THE PROBLEM

We consider a three-dimensional multilayer system composed of one elastic solid layer sandwiched between two acoustic fluid layers. Let $\mathbf{R}(\mathrm{O}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be the Cartesian frame of reference where O is the origin of the space and $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is an orthonormal basis for this space. The coordinate of the generic point $\mathbf{x}$ in $\mathbf{R}$ is specified by (x_1, x_2, x_3). The thickness of the solid layer is denoted by d and its mass density by ρ. The acoustic fluid layers occupy an open unbounded domain. The both fluids f_1 and f_2 are supposed perfects, of respective mass densities ρ_{f_1} and ρ_{f_2}; the constant speeds of sound in each fluid are c_{f_1} and c_{f_2} respectively. The interfaces between the fluids and the solid layer are infinite planes parallel to the $(\mathbf{x}_1, \mathbf{x}_2)$-plane. The $\mathbf{x}_3$-axis is oriented downward and the origin O is located at the interface between the upper fluid f_1 and the solid layer. Therefore, we assume that the structure is a two-dimensional one and that the guided waves travel in the plane $x_2 = 0$; in the following parts, this coordinate is implicit and is omitted in the mathematical expressions. Moreover, the solid layer will be so-called plate.

The elastic plate is supposed to be anisotropic and is liable to present continuously varying properties along its thickness ($\mathbf{x}_3$-axis). These mechanical properties are represented by the stiffness fourth-order tensor $\mathbb{C} = \mathbb{C}(x_3)$ and the mass density $\rho = \rho(x_3)$.

A. System equations

1. The wave equation in the fluid f_n (for $n = 1$ or 2)

In the fluid f_n and the context of the linear acoustic theory, the linearized Euler equation and the constitutive equations are written as:

$$\begin{cases} -\dfrac{\partial p^{(n)}}{\partial x_j} = \rho_{f_n} \dfrac{\partial^2 u_j^{(n)}}{\partial t^2}, \\ p^{(n)} = K_{f_n} \mathtt{div}\ \mathbf{u^{(n)}}, \end{cases} \tag{1}$$

where $\mathbf{u}^{(n)}$ and $p^{(n)}$ respectively represent the displacement vector and the pressure in the fluid f_n; its compressibility and the speed of sound in the fluid at equilibrium are respectively K_{f_n} and $c_{f_n} = \sqrt{K_{f_n}/\rho_{f_n}}$. The operator $\mathtt{div}$ is the divergence.

The solutions of the system (1) for the fluid f_n are sought under the form:

$$\mathbf{f}_n(x_1, x_2; t) = \mathbf{A}_n(x_3) \exp \imath(k_1 x_1 + k_3^{(n)} x_3 - \omega t), \tag{2}$$

where k_1 and $k_3^{(n)}$ are the wavenumbers respectively along the $\mathbf{x}_1$-axis and $\mathbf{x}_3$-axis in the fluid f_n; the angular frequency is noted ω and $\imath$ is the imaginary unit.

We consider an incident wave reaching the plate at an angle θ_1 from the $\mathbf{x}_3$-axis in the fluid f_1. The incident displacement-field is defined in the following form, assuming that its amplitude is normalized:

$$\mathbf{u}_I^{(1)} = \begin{pmatrix} \sin\theta_1 \\ 0 \\ \cos\theta_1 \end{pmatrix} \exp \imath(k_1 x_1 + k_3^{(1)} x_3 - \omega t), \tag{3}$$

with $k_1 = (\omega/c_{f_1}) \sin\theta_1$ and $k_3^{(1)} = (\omega/c_{f_1}) \cos\theta_1$. From this, the expressions of the reflected displacement-field $\mathbf{u}_R^{(1)}$ in f_1 and of the transmitted displacement-field $\mathbf{u}_T^{(2)}$ in f_2 are deduced:

$$
\mathbf{u}_R^{(1)} = R \begin{pmatrix} \sin\theta_1 \\ 0 \\ -\cos\theta_1 \end{pmatrix} \exp \imath (k_1 x_1 - k_3^{(1)} x_3 - \omega t),
$$

$$
\mathbf{u}_T^{(2)} = T \frac{c_{f_2}}{c_{f_1}} \begin{pmatrix} \sin\theta_1 \\ 0 \\ \cos\theta_1 \end{pmatrix} \exp \imath (k_1 x_1 + k_3^{(2)} x_3 - \omega t). \tag{4}
$$

where $R = R(x_1, x_3; t)$ and $T = T(x_1, x_3; t)$ respectively represent the reflection and transmission coefficients which will be expressed explicitly in the sequel. The incident, reflected and transmitted pressure fields, respectively noted $p_I^{(1)}$, $p_R^{(1)}$ and $p_T^{(2)}$, are deduced from the expressions (3) and (4) and the second equation of the system (1):

$$
\begin{aligned}
p_I^{(1)} &= -\imath\,\omega \times \rho_{f_1} c_{f_1} \times \exp \imath (k_1 x_1 + k_3^{(1)} x_3 - \omega t), \\
p_R^{(1)} &= -\imath\,\omega \times \rho_{f_1} c_{f_1} \times R \times \exp \imath (k_1 x_1 - k_3^{(1)} x_3 - \omega t), \\
p_T^{(2)} &= -\imath\,\omega \times \rho_{f_2} c_{f_2} \times T \times \exp \imath (k_1 x_1 + k_3^{(2)} x_3 - \omega t).
\end{aligned} \tag{5}
$$

B. The wave equation in the plate waveguide

The body forces in the solid plate are neglected. The balance equation of linear momentum associated with the constitutive law of linear elasticity (Hooke's law) gives the following equations:

$$
\begin{cases}
\dfrac{\partial \sigma_{ij}}{\partial x_j} = \rho \dfrac{\partial^2 u_i}{\partial t^2}, \\
\sigma_{ij} = \dfrac{1}{2} C_{ijk\ell} \left(\dfrac{\partial u_k}{\partial x_\ell} + \dfrac{\partial u_\ell}{\partial x_k} \right)
\end{cases} \tag{6}
$$

where u_i (for $i = 1, ..., 3$) and σ_{ij} (for $i, j = 1, ..., 3$) respectively represent the components of the displacement-field $\mathbf{u}$ and of the stress $\boldsymbol{\sigma}$. In the system (6), the Einstein convention of summation on repeated indices is used. The solutions are sought for the vectors of displacement $\mathbf{u}$ and traction σ_{i3} (for $i = 1, ..., 3$) (assumed to be harmonic in time t and space along the $\mathbf{x}_1$-axis) under the form:

$$
\mathbf{f}(x_1, x_3; t) = \mathbf{A}(x_3) \exp \imath (k_1 x_1 - \omega t), \tag{7}
$$

C. Fluid-loading interface conditions

The conditions at both interfaces $x_3 = 0$ and $x_3 = d$ are the continuity of the normal displacement and the one of the normal stress. We consider the fluids f_1 and f_2 are perfect, consequently, the shear stresses are zero at the interfaces ($\sigma_{13}(x_1, 0; t) = \sigma_{13}(x_1, d; t) = 0$ and $\sigma_{23}(x_1, 0; t) = \sigma_{23}(x_1, d; t) = 0$). The following relations are obtained:

$$\begin{cases} u_3(x_1, 0; t) = u_3^{(1)}(x_1, 0; t), \quad u_3(x_1, d;\ t) = u_3^{(2)}(x_1, d; t), \\ \sigma_{33}(x_1, 0; t) = -p^{(1)}(x_1, 0; t), \quad \sigma_{33}(x_1, d; t) = -p^{(2)}(x_1, d; t). \end{cases} \tag{8}$$

with

$$u_3^{(1)} = \mathbf{u}_I^{(1)}.\mathbf{x}_3 + \mathbf{u}_R^{(1)}.\mathbf{x}_3, \quad u_3^{(2)} = \mathbf{u}_T^{(2)}.\mathbf{x}_3 \quad \text{and} \quad p^{(1)} = p_I^{(1)} + p_R^{(1)}, \quad p^{(2)} = p_T^{(1)}. \tag{9}$$

D. A closed-form solution: the matricant

Introducing the expression (7) in the equation (6), we obtain the wave equation under the form of a second-order differential equation with non-constant coefficients. For particular forms of geometrical configurations, this equation has analytical solutions expressed with special functions (Bessel or Hankel functions) (Vlasie-Belloncle and Rousseau, 2003). But, in the general case, there is no analytical solution to the problem thus formulated. The most current methods to solve the wave equation in unidirectionally heterogeneous media are derived from the Thomson-Haskell method (Haskell, 1953; Thomson, 1950). These methods are appropriate for multilayered media (Hosten and Castaings, 2003; Kenneth, 1982; Levesque and Piche, 1992; Wang and Rokhlin, 2001). But, for continuously varying media, these techniques mean to replace the continuous profiles of properties by step-wise functions. Thereby, the studied problem becomes an approximate one, even before the resolution step; the accuracy of the solution as its validity domain are hard to evaluate. Moreover, the multilayered model of the waveguide creates some "virtual" interfaces likely to induce artefacts. In order to deal with the exact problem, that is to keep the continuity of the properties variation, the wave equation is re-written under the form of an ordinary differential equations system with non-constant coefficients for which an analytical solution exists: the matricant (Baron, 2005).

a. Hamiltonian form of the wave equation. We consider that the plate presents material symmetries which allow to decouple the P-SV (Pressure - Shear Vertical) waves, polarized in the propagation plane $(\mathbf{x}_1, \mathbf{x}_3)$ and the SH (Shear Horizontal) waves polarized along $\mathbf{x}_2$-axis. The incident media f_1 is a perfect fluid, only the P-SV waves travel in the plate. Applying a spatio-temporal Fourier transform on (x_1, t) of the displacement field (noted $\hat{\mathbf{u}}(k_1, x_3; \omega)$ after Fourier transform) and on the traction field (noted $\hat{\sigma}_{i3}(k_1, x_3; \omega)$ for $i = 1, ..., 3$) and using the Voigt notation ($C_{ijk\ell}$ for $i, j, k, \ell = 1, ..., 3$ is replaced by c_{IJ} for $I, J = 1, ..., 6$), Eq. (6) leads to:

$$\frac{\partial \hat{\sigma}_{13}}{\partial x_3} = \rho(\imath\,\omega)^2 \hat{u}_1 - \imath k_1 \hat{\sigma}_{11}, \quad \frac{\partial \hat{\sigma}_{33}}{\partial x_3} = \rho(\imath\,\omega)^2 \hat{u}_3 - \imath k_1 \hat{\sigma}_{13}, \tag{10}$$

$$\hat{\sigma}_{11} = \imath k_1 c_{11} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}, \qquad \hat{\sigma}_{13} = c_{55}\left(\frac{\partial \hat{u}_1}{\partial x_3} + \imath k_1 \hat{u}_3\right), \qquad \hat{\sigma}_{33} = \imath k_1 c_{13} \hat{u}_1 + c_{33} \frac{\partial \hat{u}_3}{\partial x_3}. \tag{11}$$

According to Eqs. (10) and (11), $\hat{\sigma}_{11}$ is function of $\hat{u}_1$ and $\hat{\sigma}_{33}$. The wave equation becomes a matrix system expressed using the Thomson-Haskell parametrization of the Stroh formalism (Stroh, 1962):

$$\frac{d}{dx_3} \boldsymbol{\eta}(x_3) = \imath\,\omega \mathbf{Q}(x_3) \boldsymbol{\eta}(x_3), \tag{12}$$

that is:

$$\frac{d}{dx_3} \begin{pmatrix} \imath\,\omega \hat{u}_1 \\ \imath\,\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix} = \imath\omega \begin{pmatrix} 0 & s_1 & 1/c_{55}(x_3) & 0 \\ -s_1 c_{13}(x_3)/c_{33}(x_3) & 0 & 0 & 1/c_{33}(x_3) \\ \rho(x_3) - s_1^2 \zeta(x_3) & 0 & 0 & -s_1 c_{13}(x_3)/c_{33}(x_3) \\ 0 & \rho(x_3) & -s_1 & 0 \end{pmatrix} \begin{pmatrix} \imath\,\omega \hat{u}_1 \\ \imath\,\omega \hat{u}_3 \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{33} \end{pmatrix}, \tag{13}$$

with the relations:

$$\zeta(x_3) = c_{11}(x_3) - \frac{c_{13}^2(x_3)}{c_{33}(x_3)}, \quad k_1 = \omega s_1, \tag{14}$$

where s_1 is the x_1-component of the slowness. The matrix $\mathbf{Q}$ includes all the information about the heterogeneity of the waveguide because it is expressed from the plate mechanical properties $(\rho(x_3), \mathbb{C}(x_3))$ and from two acoustical parameters (s_1, ω).

b. Explicit solution: the Peano expansion of the matricant. The wave equation thus formulated has an analytical solution expressed between a reference point $(x_1, 0, x_3^0)$ and some point of the plate $(x_1, 0, x_3)$ in the propagation plane. This solution is called the matricant and is explicitly written under the form of the Peano series expansion (Gantmacher, 1959; Peano, 1888; Pease, 1965):

$$\mathbf{M}(x_3, x_3^0) = \mathbf{I} + (\imath\,\omega)\int_{x_3^0}^{x_3} \mathbf{Q}(\xi)d\xi + (\imath\,\omega)^2\int_{x_3^0}^{x_3} \mathbf{Q}(\xi)\Big(\int_{x_3^0}^{\xi} \mathbf{Q}(\xi_1)d\xi_1\Big)d\xi + ..., \qquad (15)$$

where $\mathbf{I}$ is the identity matrix of dimension $(4, 4)$. If the matrix components $\mathbf{Q}(x_3)$ are bounded in the study interval, these series are always convergent (Baron, 2005). The components of the matrix $\mathbf{Q}$ are continuous in x_3 and the study interval is bounded (thickness of the waveguide), consequently the hypothesis is always verified. We underline that the $\imath\omega$-factorization leads up to a polynomial form of the matricant. The $\imath\omega$-polynomial coefficients are matrices independent of ω.

c. Boundary conditions: fluid-structure interaction. Using the propagator property of the matricant through the plate thickness, the state-vector (defined in (13)) at the second interface $\boldsymbol{\eta}(d)$ is evaluated from the state-vector at the first interface $\boldsymbol{\eta}(0)$ as follows:

$$\boldsymbol{\eta}(d) = \mathbf{M}(d, 0)\boldsymbol{\eta}(0). \qquad (16)$$

The fluid-structure interaction sets the conditions of zero shear stresses (see subsection III C), used after a spatio-temporal Fourier transform on (x_1, t). The equation (16) becomes:

$$\begin{pmatrix} \imath\,\omega\hat{u}_1(k_1, d; \omega) \\ \imath\,\omega\hat{u}_3(k_1, d; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, d; \omega) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{pmatrix} \begin{pmatrix} \imath\,\omega\hat{u}_1(k_1, 0; \omega) \\ \imath\,\omega\hat{u}_3(k_1, 0; \omega) \\ 0 \\ \hat{\sigma}_{33}(k_1, 0; \omega) \end{pmatrix}. \qquad (17)$$

The condition to obtain a nontrivial solution to the equation (17) leads to the following relation:

$$\imath\,\omega\hat{u}_1(k_1, 0; \omega) \times M_{13} + \imath\,\omega\hat{u}_3(k_1, 0; \omega) \times M_{32} + \hat{\sigma}_{33}(k_1, 0; \omega) \times M_{34} = 0, \qquad (18)$$

where M_{ij} (for $i, j = 1, ..., 4$) represent the components of the matrix $\mathbf{M}$. The displacement component $\hat{u}_1(k_1, 0; \omega)$ can be expressed as a linear combination of $\hat{u}_3(k_1, 0; \omega)$ and $\hat{\sigma}_{33}(k_1, 0; \omega)$; thus the system (16) of dimension 4 is reduced to a matrix system of dimension 2:

$$\boldsymbol{\eta}(d) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \boldsymbol{\eta}(0), \qquad \text{where } \boldsymbol{\eta}(x_3) = \begin{pmatrix} \imath\,\omega \hat{u}_3 \\ \hat{\sigma}_{33} \end{pmatrix}. \tag{19}$$

with the relations:

$$\begin{aligned} P_1 &= M_{22} - M_{21}\frac{M_{32}}{M_{31}}, \quad P_2 = M_{24} - M_{21}\frac{M_{34}}{M_{31}}, \\ P_3 &= M_{42} - M_{41}\frac{M_{32}}{M_{31}}, \quad P_4 = M_{44} - M_{11}\frac{M_{34}}{M_{31}}. \end{aligned} \tag{20}$$

The interface conditions (8) are transformed in the Fourier domain (k_1, ω). The expressions of the displacement and the pressure in the fluids (see Eqs. (3), (4) and (5), so that the one of the displacement and traction fields in the solid plate (see Eq. (19), are substituted in the transformed interface conditions. Setting $\boldsymbol{\eta}(0) = (\alpha_1, \alpha_2)^T \exp \imath(k_1 x_1 - \omega t)$, where the superscript $.^T$ designates the transpose operator, we obtain the following matrix equation:

$$\begin{pmatrix} \imath\,\omega s_3^{(1)} c_{f_1} & 1 & 0 & 0 \\ -\imath\,\omega \rho_{f_1} c_{f_1} & 0 & 1 & 0 \\ 0 & P_1 & P_2 & -\imath\,\omega s_3^{(2)} c_{f_2} \exp\left(\imath\,\omega s_3^{(2)} d\right) \\ 0 & P_3 & P_4 & -\imath\,\omega \rho_{f_2} c_{f_2} \exp\left(\imath\,\omega s_3^{(2)} d\right) \end{pmatrix} \begin{pmatrix} \hat{R} \\ \alpha_1 \\ \alpha_2 \\ \hat{T} \end{pmatrix} = \begin{pmatrix} \imath\,\omega s_3^{(1)} c_{f_1} \\ \imath\,\omega \rho_{f_1} c_{f_1} \\ 0 \\ 0 \end{pmatrix}, \tag{21}$$

where $\mathbf{s}^{(n)} = \mathbf{k}^{(n)}/\omega$ is the slowness-vector in the fluid f_n ($n = 1$ or 2). The quantities $\hat{R}$ and $\hat{T}$ are respectively the reflection and transmission coefficients expressed in the Fourier domain: $\hat{R} = \hat{R}(k_1, x_3; t)$ and $\hat{T} = \hat{T}(k_1, x_3; t)$. The two first lines of system (21) express the boundary conditions at the first interface ($x_3 = 0$) and the two last lines those at the second interface ($x_3 = d$) introducing the Fourier transform of the expressions (3), (4) and (5) in the following relations:

$$\boldsymbol{\eta}(0) - \begin{pmatrix} \imath\,\omega \hat{u}_{3_R}^{(1)} \\ -\hat{p}_R \end{pmatrix}_{x_3=0} = \begin{pmatrix} \imath\,\omega \hat{u}_{3_I}^{(1)} \\ -\hat{p}_I \end{pmatrix}_{x_3=0}, \qquad \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \boldsymbol{\eta}(0) - \begin{pmatrix} \imath\omega \hat{u}_{3_T}^{(2)} \\ -\hat{p}_T \end{pmatrix}_{x_3=d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{22}$$

where $\hat{u}_{3_R}^{(1)}, \hat{u}_{3_I}^{(1)}$ and $\hat{u}_{3_T}^{(2)}$ are the components along $\mathbf{x}_3$-axis of $\hat{\mathbf{u}}_R^{(1)}, \hat{\mathbf{u}}_I^{(1)}$ and $\hat{\mathbf{u}}_T^{(2)}$ vectors respectively. Note the equality between the quantities $u_{3_T}^{(2)}$ and $u_3^{(2)}$ where this last is defined in Eq. (9).

d. Expression of the reflection and transmission coefficients. From (21), we deduce the analytical expressions of the reflection and transmission complex coefficients:

$$\begin{aligned}\hat{R}(s_1, x_3; \omega) &= \frac{(P_3 - P_1 Z_2 + P_4 Z_1 - P_2 Z_1 Z_2)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)}, \\ \hat{T}(s_1, x_3; \omega) &= -\frac{2 Z_2 (\rho_{f_1} c_{f_1} / \rho_{f_2} c_{f_2})(P_1 P_4 - P_2 P_3)}{(P_3 - P_1 Z_2 - P_4 Z_1 + P_2 Z_1 Z_2)} \exp(-\imath \omega s_3^{(2)} d),\end{aligned} \tag{23}$$

with $Z_n = \rho_{f_n} / \sqrt{1/c_{f_n}^2 - s_1^2}$ (for $n = 1$ or 2).

IV. VALIDATION OF THE METHOD

The aim of this section is to check that the Peano expansion of the matricant is well-adapted to study fluid-loaded waveguides. We take into account the fluid-structure interaction in different configurations of homogeneous plates comparing the results obtained from the numerical implementation of the Peano expansion of the matricant to results taken from the literature.

The numerical evaluation of P_1, P_2, P_3 and P_4 requires us to truncate the Peano series and to numerically calculate the integrals. Thus, the error can be estimated and controlled (Baron, 2005). We retained 70 terms in the series and evaluate the integrals over 100 points using the Simpson's rule (fourth-order integration method). These choices ensure the convergence of the solution and the accuracy of the results for a reasonable computation time (never exceeding few minutes on a desktop computer). The expressions (23) give the frequency spectrum (*modulus* and phase) of the reflection coefficient for different incidences (s_1 varies from zero –normal incidence– to $1/c_{f_1}$ corresponding to the critical incidence in the fluid f_1). A lot of works detailed the relationship between the poles and the zeros of the reflection coefficient and the leaky Lamb waves dispersion curves (Chimenti and Rokhlin, 1990; Deschamps and Poncelet, 2000).

The results of sub-section IV A compare the dispersion curves obtained by seeking the poles of the reflection coefficient (23) and the results taken from the literature or from closed-form solution.

A. Validation for a homogeneous and isotropic or anisotropic fluid-loaded plate

The method is tested by plotting the dispersion curves (variation of the phase velocity *versus* frequency-thickness product) for an isotropic aluminium plate immersed in water. The data in the paper of Chimenti and Rokhlin (1990) are used. The results obtained (not shown) by the present method are in perfect agreement with the results presented by them (Chimenti and Rokhlin, 1990).

As mentioned by Chimenti and Rokhlin (1990), there are few differences between the zeros *loci* and the poles *loci* for a plate immersed in a fluid whose the mass density is lower than the plate mass density. As underlined by several authors, fluid-load does have just a weak influence on guided wave travelling in the plate immersed in water.

Taking into account the anisotropy does not change the scheme for the numerical solution of wave equation with the matricant. We consider a transverse isotropic plate immersed in water ($\rho_f = 1$ g.cm^{-3}, $c_f = 1.485$ mm.μs^{-1}) whose properties are reported in Tab. I. For that configuration, Nayfeh and Chimenti (1989) developed a method to obtain an analytical expression of the reflection coefficient. By using the data from this paper, the results obtained (see Fig. 1) with the present method are in perfect agreement with theirs. The curves presented in Fig. 1.a and Fig. 1.b are superimposed and need to be presented separately.

B. Validation for a asymmetrically loaded homogeneous isotropic plate ($f_1 \neq f_2$)

The formalism presented here to solve the wave equation in an unidirectionally graded medium presents two main advantages: without changing the scheme to obtain the numerical solution we can take into account i) an asymmetric loading and ii) the unidirectional continuous heterogeneity.

The mechanical behavior of the plate is different for symmetric and asymmetric loadings. For example, in the symmetric loading case, there is a unique critical frequency and a unique phase velocity value v_φ in the plate, which corresponds to the propagation velocity in the fluid ($\mathrm{v}_\varphi = c_{f_1} = c_{f_2}$), for which the displacements and the stresses at the interfaces are quasi-null; whereas in the asymmetric loading ($f_1 \neq f_2$), there are two critical frequencies and two values of the phase velocity in the plate for which the structure does not respond (Dickey *et al.*, 1995). The validation is done on an isotropic aluminium plate with the following properties: $\rho = 2.79$ g.cm^{-3} ; the longitudinal and transverse waves velocities are respectively $\mathrm{v}_L = 6.38$ mm.μs^{-1} and $\mathrm{v}_T = 3.10$ mm.μs^{-1}. The characteristic properties of the fluid f_1 correspond to those of water (see sub-section IV A); the characteristic properties of the fluid f_2 correspond to glycerine: $\rho_{f_2} = 1.26$ g.cm^{-3} and $c_{f_2} = 1.920$ mm.μs^{-1}. This configuration is the same as the one studied by Franklin *et al.* (2001). The *modulus* of the reflection coefficient *versus* the incident angle is plotted in the Fig. 2 for a fixed frequency-thickness product ($f \times \ d = 4.7$ MHz.mm).

This figure shows the perfect agreement between our results and the ones presented by Franklin *et al.* (2001).

V. RELEVANCY OF THE METHOD FOR ULTRASOUND CHARACTERIZATION OF CORTICAL BONE

Cortical bone is a kind of hard tissue found at the edges of long bones and supports most of the load of the body. Several studies demonstrated the heterogeneous nature of the cortical bone, particularly they show evidence the gradual variation of the volumetric porosity (ratio between pores and total volume) along the cortical thickness. Yet, the porosity is intrinsically linked to the macroscopic mechanical behavior of the cortical bone (Baron *et al.*, 2007). Therefore, the continuous variation of porosity induces a continuous variation of material properties. Taking into account the gradient should prove itself to be essential in the context of diagnosis and therapeutic monitoring of osteoporosis. Indeed, the gradient characterization would allow to assess geometrical (cortex thickness) and material (elastic coefficients variation) information, which are fundamental parameters to evaluate the bone fragility. For several years, the quantitative ultrasonography (by axial and transverse transmissions) proved itself to be an alternative hopeful technique to evaluate the fracture risk (Marin *et al.*, 2006). However, the inter-individual and inter-site variations of bone mechanical properties make the standardization of the protocol of fracture risk evaluation by ultrasound very delicate.

The focus is set on a configuration closed to the axial transmission device for *in vivo* conditions. In this context, the relevancy of studying the lateral wave propagation has been demonstrated (Bossy *et al.*, 2004a,b; Camus *et al.*, 2000). As a consequence all the reflection coefficient presented in this paper were calculated for an incident angle corresponding to the grazing-angle for longitudinal waves (critical angle of longitudinal waves propagation in the plate at the first interface ($x_3 = 0$)).

The surrounding media in the *in vivo* configuration of ultrasound characterization of cortical bone are the muscle for the upper fluid f_1 ($c_{f_1} = 1.54$ mm.μs and $\rho_{f_1} = 1.07$ g.cm^{-3}) and the marrow for the lower fluid f_2 ($c_{f_2} = 1.45$ mm.μs and $\rho_{f_2} = 0.9$ g.cm^{-3}) (Burlew *et al.*, 1980; Hill *et al.*, 1986). We are interested in the influence of the continuous gradient of the mechanical properties on the ultrasonic response in the configuration of *in vivo* cortical bone characterization.

A. Determination of a realistic range of variation of elastic bone properties

In order to define numerical values for a realistic value of the gradient of the different material properties, it is necessary to determine the limiting values reached by each elastic property. Our approach consists in considering *in vitro* measurements published in Dong and Guo (2004) and performed in 18 samples. It is assumed that these limiting values for elastic properties are relevant for physiologic ranges of variations. Furthermore, the elastic coefficients of the stiffness tensor are constrained to fully verify the thermodynamical conditions of stability (see Appendix).

We assume that cortical bone is transverse isotropic. Transverse isotropy has been shown experimentally by different authors (Dong and Guo, 2004; Reilly and Burnstein, 1974; Rho, 1996) to be a realistic approximation of cortical bone degree of anisotropy.

Dong and Guo (2004) measured the homogenized bone properties by performing tensile and torsional tests with a mechanical testing system on 18 different human femoral bone specimens. The authors measured the values of the longitudinal and transverse Young's moduli (E_L and E_T respectively) as well as the values of the longitudinal shear *modulus* G_L. From these measurements and by assuming constant values of Poisson's ratio, the values of the different components of the stiffness tensor corresponding to the values of E_L, E_T and G_L measured in Dong and Guo (2004) were obtained following the relationships given in Appendix.

The value of the longitudinal Poisson's ratio ν_L is taken equal to 0.37 for all computations because it corresponds to the average value found in Dong and Guo (2004). The value of the transverse Poisson's ratio ν_T is taken equal to 0.45 following Eqs. (A.3) of Appendix. The values of the stiffness coefficients corresponding to the mean values of the bone mechanical properties are referred to as 'reference' set of parameters in what follows. The maximum and minimum values of the stiffness coefficients are obtained by considering respectively the maximum and minimum values of E_L and E_T within the range of variation measured in Dong and Guo (2004), which is a simple way to obtain a realistic range of variation for the stiffness coefficients in cortical bone. Furthermore, the elastic properties deduced from the approach reported above were constrained to verify the thermodynamical stability conditions given in the Appendix by Eqs. (A.4).

We choose a mean value of mass density ρ equal to 1.722 g.cm^{-3}, following the value taken in Macocco *et al.* (2006). This value is chosen for the reference mass density. In order to derive a realistic range of variation for mass density, we assume that the reference value is given by a porosity of 7%, which correspond approximately to the mean porosity at the radius (Baron *et al.*, 2007). The porosity was assumed to vary between 3 and 15% (Bousson *et al.*, 2001; Dong and Guo, 2004) and a rule of mixture leads to the range of variation of mass density.

B. Modeling a gradient of material property

The impact of a controlled gradient vector $\boldsymbol{\delta}$ of any investigated material property S on the response of the structure studied is assessed. The scalar S corresponds to one of the stiffness coefficients c_{ij} of $\mathbb{C}$ or to mass density ρ. In each set of simulations, all the material properties are constant and equal to their reference value while S is subjected to the defined gradient.

The gradient vector $\boldsymbol{\delta} = \texttt{grad}\, S = \delta\, \mathbf{x}_3$ is assumed to be independent of x_1 in all cases, where $\mathbf{x}_3$ is an unit vector along $\mathbf{x}_3$-axis and $\texttt{grad}$ is the gradient operator acting on a scalar field. The quantity δ is always taken positive because the porosity is known to be higher in the endosteal part than in the periosteal part of the bone. Moreover, only the simple situation of affine spatial variations of S is considered, corresponding to a constant value of δ. This affine spatial variation of S is chosen because the actual physiological spatial dependence of S remains unknown. Two different affine spatial dependencies of the studied material property are considered and are illustrated in Fig. 3. Associated gradient δ will be referred to as type 1 or 2.

Type 1. The gradient of type 1 is such that the physical property S takes the same value S_M at the upper interface $x_3 = 0$ of the solid plate for all values of the gradient δ. The quantity $S(x_3)$ is therefore given by:

$$S(x_3) = S_M + \delta \times x_3, \tag{24}$$

where S_M is given by the maximal value of the material property S considered. The maximal value δ_M of δ is chosen so that $S(d)$ is equal to S_m, where S_m is given by the minimal value of S. The gradient δ_M is given by:

$$\delta_M = \frac{(S_m - S_M)}{d}. \tag{25}$$

Type 2. The gradient of type 2 is such that the physical property S takes the same value at the middle $x_3 = d/2$ of the solid plate for all values of gradient δ. Furthermore, the mean value of the property S is identical for all δ. The quantity $S(x_3)$ is given by:

$$S(x_3) = \frac{(S_m + S_M)}{2} + \delta \times (x_3 - \frac{d}{2}). \tag{26}$$

The maximal value of δ is also given by Eq. (25), so that all values of $S(x_3)$ are again always comprised between S_m and S_M. Again, the maximal value δ_M of δ is given by Eq. (25).

Gradient of type 2 leads for all magnitudes of δ to a constant value of the spatial average of the gradient.

For both types of spatial variations, five different values of δ regularly distributed between 0 and δ_M are arbitrarily considered in the thickness.

Table II recalls the maximum, minimum and mean measured values of E_L, E_T and G_L as given by Dong and Guo (2004). Table II also shows the maximum, minimum and mean values of the 4 components (c_{11}, c_{13}, c_{33} and c_{55}) of the stiffness tensor $\mathbb{C}$ affecting wave propagation derived from Eqs. (A.3) of Appendix.

In Table III, the minimal and maximal values of each variable corresponding to the realistic range of variation obtained i) by considering the reference values of Table II and

ii) by verifying that the thermodynamical stability conditions are fulfilled. Values resulting from the stability conditions are marked with an asterisk.

In the simulations, the values of S_M and S_m are those reported in Table III.

C. Results et discussion

First of all, the method allows to investigate the influence of the fluids on the ultrasonic response. In the case of the characterization of cortical bone, the two fluids f_1 and f_2 are different which correspond to an asymetrical loading (see sub-section IV B): the fluid f_1 has been considered as muscle ($c_{f_1} = 1\ 540$ m.s^{-1} and $\rho_{f_1} = 1.07$ g.cm^{-3}) and fluid f_2 as marrow ($c_{f_2} = 1\ 450$ m.s^{-1} and $\rho_{f_2} = 0.9$ g.cm^{-3}). The properties of these two fluids are very closed to those of water. The frequency spectrum of the reflection coefficient *modulus* has been plotted for the *in vivo* configuration and compared with the result obtained for a cortical bone plate immersed in water for the ten profiles of mechanical properties (Figure not shown). For homogeneous plates as for linearly graded plates, the two curves are very closed however the *modulus* of the reflection coefficient at null-frequency is not null and the minimum values are greater than for water but obtained for the same frequency-thickness products. Thats is why, all the following results have been calculated for a cortical bone plate immerged in water.

The reflection coefficient calculated with the Peano series of the matricant is sensitive to the variation of the properties gradient. As we consider that the osteoporosis entails a trabecularization of cortical bone from the endosteal side, the characterization of the gradient of the properties between the endosteal and periosteal regions may be an element of the diagnosis of the osteoporosis progress and of the therapeutic follow-up.

It is known that the gradient of the properties along the cortical thickness is due to the continuous variation of the porosity growing progressively from the periosteal to the endosteal regions. From previous work, we know that the porosity influence all the stiffness coefficients (Baron *et al.*, 2007). The frequency spectrum of the reflection coefficient has been plotted for the ten profiles presented on Fig. 3 applied to all the stiffness coefficients implied (c_{11}, c_{13}, c_{33} and c_{55} and to the mass density ρ) (see Table III). The reflection coefficients have been calculated at an incident angle corresponding to the grazing-angle (critical angle for the longitudinal waves in the bone plate). For the two types of gradients, differences appear between all the gradients and the homogeneized plates (corresponding to the maximum value for Type 1 and to the average value for Type 2) particularly on the location of the *extrema* values of the reflection coefficient *modulus* (see Fig. 4).

The increase of the gradient of properties shifts the minimum and maximum values forward high frequency-thickness products. However, the results (see Fig. 4) put on evidence that the behavior of the reflection coefficient *modulus* is sensibly the same for frequency-thickness products between 0 and 1.5 MHz.mm. Beyond this value, the behavior is clearly different. At sufficiently high frequency, the wavelength is smaller and more sensitive to the affine variation of the material properties. It is noteworthy that for a heterogeneous waveguide, the *minima* of the reflection coefficient magnitude do not reach zero (except for null frequency) but end at a finite value and the changes in phase (not shown) are not so rapid, which means that total transmission does not take place in this situation.

The influence of the variation of each parameter (stiffness coefficients) on the frequency spectrum of the reflection coefficient has been investigated. This analysis has been lead for an incidence angle corresponding to the longitudinal waves critical angle in the plate at $x_3 = 0$. This incidence corresponds to the generation and the propagation of the lateral wave.

It appears that each of them has an impact on the reflection wave, but the leading term is c_{11}. The frequency spectrum of the reflection coefficient for a varying c_{11} in an affine way is very closed to the frequency spectrum of the reflection coefficient obtained for the affine variation of all the material properties (c_{ij} and ρ) and is the most different to the results from homogeneized plates (average value or maximum value) compared to the frequency spectrum calculated for the *one-parameter variation* of the other elastic parameters (c_{13}, c_{33} and c_{55} and ρ) (see Fig. 5). It is noteworthy that c_{11} is the stiffness coefficient associated with the axial direction and determining the speed of the lateral wave. So, it seems that the lateral wave would be the indicator of the c_{11} gradient. It is important to note that for homogeneized plates (extremum value or average) the frequency spectrum of the reflection coefficient is really different from that of the plate with continuously varying properties. We infer that the approximation by an homogeneous plate of cortical bone and all the more so for the osteoporotic cortical bone (for which the gradient would be greater) may induce bad interpretation of the ultrasonic response.

The authors do not know any results in the literature about the measurement of the variation of the porosity within the cortical thickness. The assumption of a affine gradient is a first step, others gradients may be investigated and the method presented in this article would be applied in the same way for non-affine gradients (Shuvalov *et al.*, 2005).

VI. CONCLUSION AND PERSPECTIVES

The sextic plate formalism has been employed for analyzing the leaky Lamb waves in anisotropic heterogeneous plates immersed in fluids. This formalism and especially the polynomial form of the solution (see Eq. (15)) presents several analytical and numerical advantages. First, the low-frequency asymptotics are naturally assessed evaluating only two or three terms in the series (Shuvalov *et al.*, 2005). The information thus collected are of major interest in the analysis of the elastic behavior of waveguides (Baron *et al.*, 2008). Secondly, the polynomial-form makes the numerical evaluation of the solution faster. Indeed, the polynomial coefficients are independent of the frequency, so they are calculated for a fixed slowness value and stored. When the frequency varies, there is no need to re-calculate the polynomial coefficients, it comes to a polynomial evaluation whose coefficients are perfectly known, which is time-saving.

The Peano series of the matricant is a method which keeps the continuity of the profiles and so, the authenticity of the problem. One of the key points for methods based on multilayered media to deal with FGM is to relevantly discretize the properties profiles. The choice of the dicretization may lead to some errors especially in the evaluation of the resonances.

This elegant mathematical tool is also very adaptative to different physical problems. In the case studied –a fluid-loaded plane waveguide– the anisotropy, the heterogeneity (continuous or discontinuous variation of properties) and the asymmetric fluid-loading are taken into account without changing the resolution scheme.

Further work needs to be done to relate the results presented in this paper to dispersion curves and propagation of transient and heterogeneous waves in a fluid-loaded continuously heterogeneous waveguide.

Furthermore, from this study, the transient response of a fluid-loaded plate is considered. The frequency spectrum of the reflection coefficient is calculated for incidences between the normal and critical incidences for compression waves in the fluid f_1. Thus, the plate transfer function is calculated in the Fourier domain (x_1-wavenumber, frequency): $\hat{R}(k_1, x_3; \omega)$. A double inverse Fourier transform on (k_1, ω) is applied on $\hat{R}(k_1, x_3; \omega)$ to transform into the space-time domain; the temporal signals can be obtained at different points along the propagation $\mathbf{x}_3$-axis: $R(x_1, x_3; t)$.

Lastly, the formalism presented here is well-adapted to deal with wave propagation in anisotropic tubes with radial property gradients (Shuvalov, 2003). The wave equation keeps the same form as Eq. (12), the state vector is expressed from the displacement and traction components in the cylindrical basis and the matrix $\mathbf{Q}$ depends on the radial position r ($\mathbf{Q} = \mathbf{Q}(r)$). In cylindrical homogeneous structures, taking into account an anisotropy more important than transverse isotropy is fussy because there is no analytical solution to the "classical" wave equation (second-order differential equation). The Stroh's formalism (hamiltonian formulation of the wave equation) (Stroh, 1962), upon which the Peano expansion of the matricant is based, is a promising alternative solution which allows to consider altogether the geometry (cylinder), the anisotropy and the heterogeneity (radial property gradients) of a structure.

Appendix: THERMODYNAMICS STABILITY CONDITIONS AND STIFFNESS COEFFICIENTS

The Hooke's law is written under the form $\sigma_{ij} = C_{ijk\ell}\, \epsilon_{k\ell}$ for $(i, j, k, \ell = 1, ..., 3)$, where $\boldsymbol{\sigma}$ is the stress-tensor, $\boldsymbol{\epsilon}$ is the strain-tensor and $\mathbb{C}$ is the fourth-order stiffness-tensor. In the transversely isotropic case, with $(\mathbf{x}_2, \mathbf{x}_3)$ as isotropic plane, the stiffness-tensor is expressed as a stiffness matrix (using Voigt's notation):

$$\mathbb{C} = \begin{pmatrix} c_{11} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{pmatrix}. \tag{A.1}$$

We introduce the matrix $\mathbb{S}$, the inverse of the matrix $\mathbb{C}$. It is expressed by:

$$\mathbb{S} = \begin{pmatrix} 1/E_L & -\nu_L/E_L & -\nu_L/E_L & 0 & 0 & 0 \\ -\nu_L/E_L & 1/E_T & -\nu_T/E_T & 0 & 0 & 0 \\ -\nu_L/E_L & -\nu_T/E_T & 1/E_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_L & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_L \end{pmatrix}. \tag{A.2}$$

with $E_{L,T}$, the longitudinal (L) and transverse (T) Young's *moduli*; $\nu_{L,T}$, the longitudinal (L) and transverse (T) Poisson's ratios; and $G_{L,T}$, the longitudinal (L) and transverse (T) shear *moduli.* By inverting (A.1) and identifying it with (A.2), we obtain the following relations:

$$\begin{aligned} E_L &= \frac{c_{11}c_{33} - 2c_{13}^2 + c_{11}c_{23}}{c_{33} + c_{23}}, & \nu_L &= \frac{c_{13}}{c_{33} + c_{23}}, \\ E_T &= \frac{c_{11}(c_{33}^2 - c_{23}^2) + 2c_{13}^2(c_{23} - c_{33})}{c_{11}c_{33} - c_{13}^2}, & \nu_T &= \frac{c_{11}c_{23} - c_{13}^2}{c_{11}c_{33} - c_{13}^2}, \\ G_T &= c_{44}, & G_L &= c_{55}. \end{aligned} \tag{A.3}$$

Knowing the stiffness coefficients values, we can verify if the thermodynamical stability conditions are satisfied:

$$E_L > 0, \quad E_T > 0, \quad -1 < \nu_T < 1, \quad \frac{(1-\nu_T)}{2}\frac{E_L}{E_T} - \nu_L^2 > 0, \quad G_L > 0. \tag{A.4}$$

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Table I. Elastic properties of transversely isotropic plate (with $c_{23} = c_{22} - 2c_{44}$).

ρ (g.cm^{-3})	c_{11} (GPa)	$c_{22} = c_{33}$ (GPa)	$c_{12} = c_{13}$ (GPa)	c_{44} (GPa)	$c_{66} = c_{55}$ (GPa)
1.85	23.05	15.1	8.7	3.25	4.7

Table II. Mean value, maximum and minimum values of the homogenized longitudinal and transversal Young *modulus*, of the four elastic constants and of mass density affecting the ultrasonic propagation in the framework of the model. These values are taken from Dong and Guo (2004).

Mechanical quantity	E_L (GPa)	E_T (GPa)	c_{11} (GPa)	c_{13} (GPa)	c_{33} (GPa)	$c_{55} = G_L$ (GPa)	ρ (g.cm^{-3})
Mean value (reference)	16.6	9.5	23.1	8.7	15.1	4.7	1.722
Minimum	13.4	6.5	17.6	5.1	9.1	3.3	1.66
Maximum	20.6	12.8	29.6	15.9	25.9	5.5	1.753

Table III. The minimal and maximal values of each variable corresponding to the realistic range of variation obtained i) by considering the reference values of Table II and ii) by verifying that the thermodynamical stability conditions are fulfilled. Values resulting from the stability conditions are marked with an asterisk.

Material property S	c_{11} (GPa)	c_{13} (GPa)	c_{33} (GPa)	$c_{55} = G_L$ (GPa)	ρ (g.cm^{-3})
Realistic range $[S_m, S_M]$ (reference)	$[17.6, 29.6]$	$[5.1, 11.1^*]$	$[11.8^*, 25.9]$	$[3.3, 5.5]$	$[1.66, 1.753]$

LIST OF FIGURES

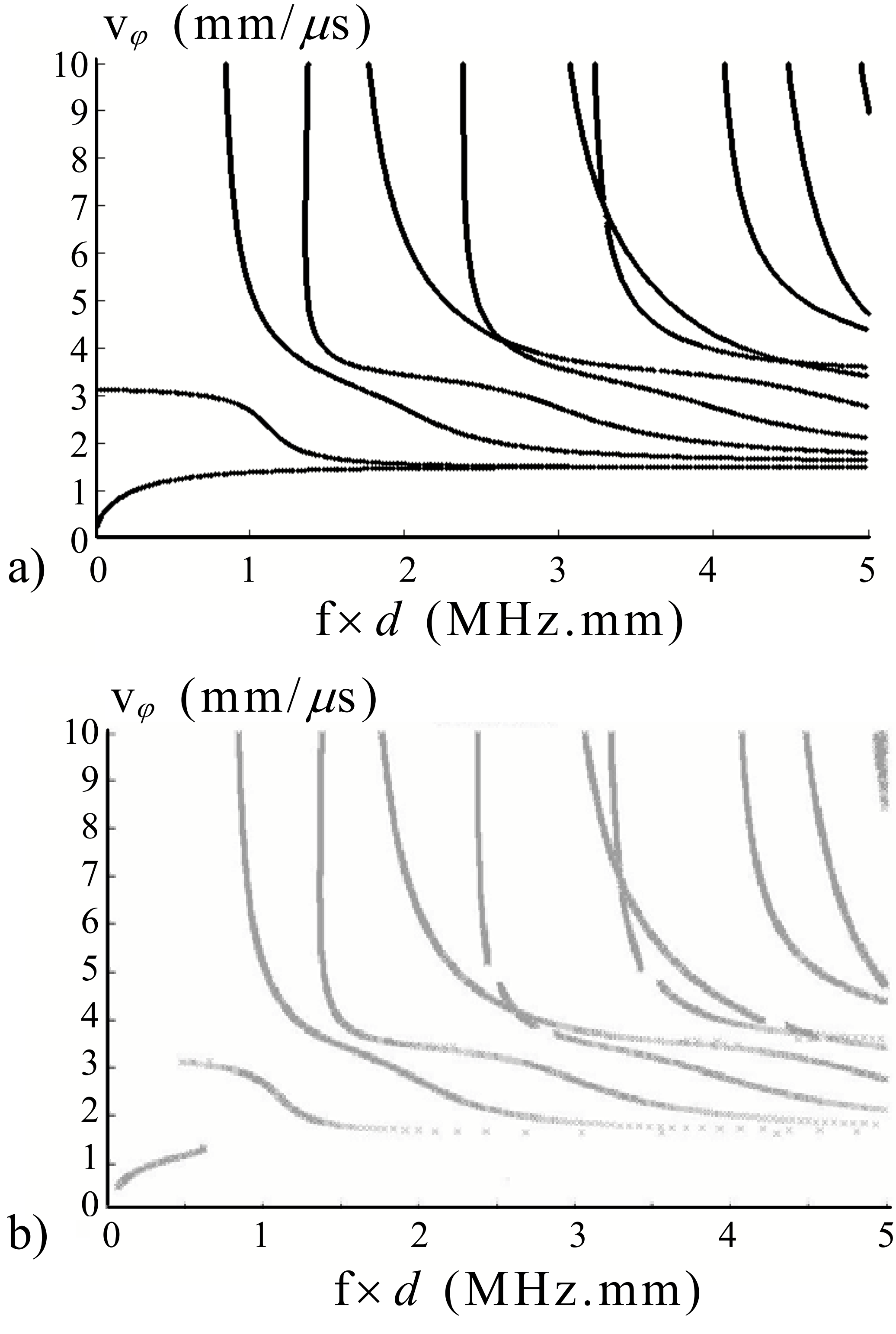
v$_\varphi$ (mm/μs)
10
9
8
7
6
5
4
3
2
1
0
a)
0
1
2
3
4
5
f× d (MHz.mm)
v$_\varphi$ (mm/μs)
10
9
8
7
6
5
4
3
2
1
0
b)
0
1
2
3
4
5
f× d (MHz.mm)

Figure 1.

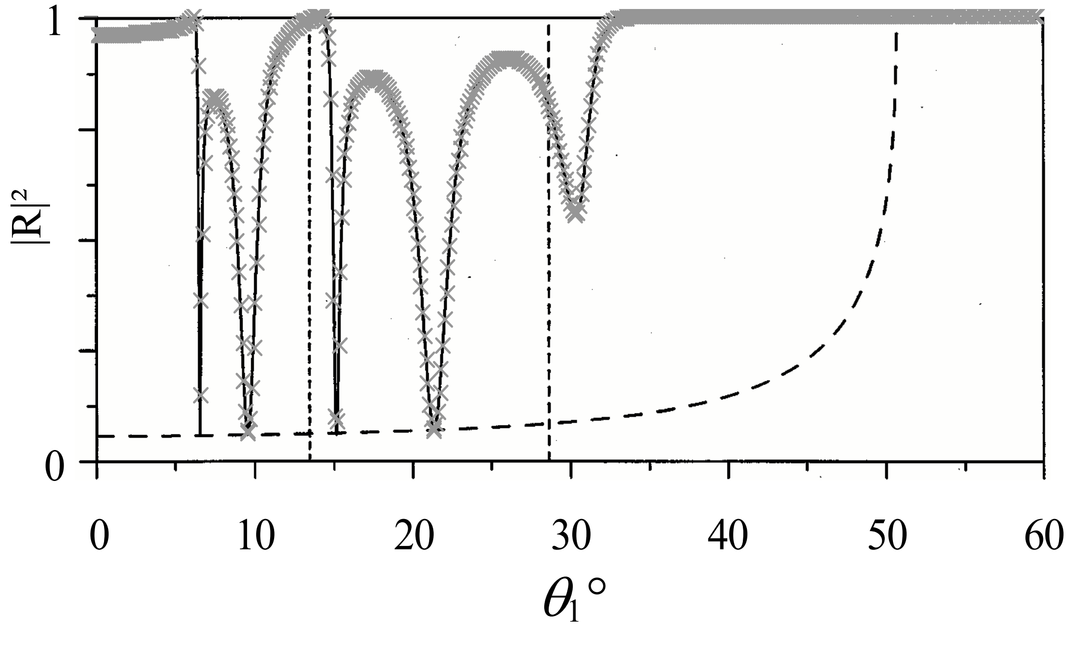

Figure 2.

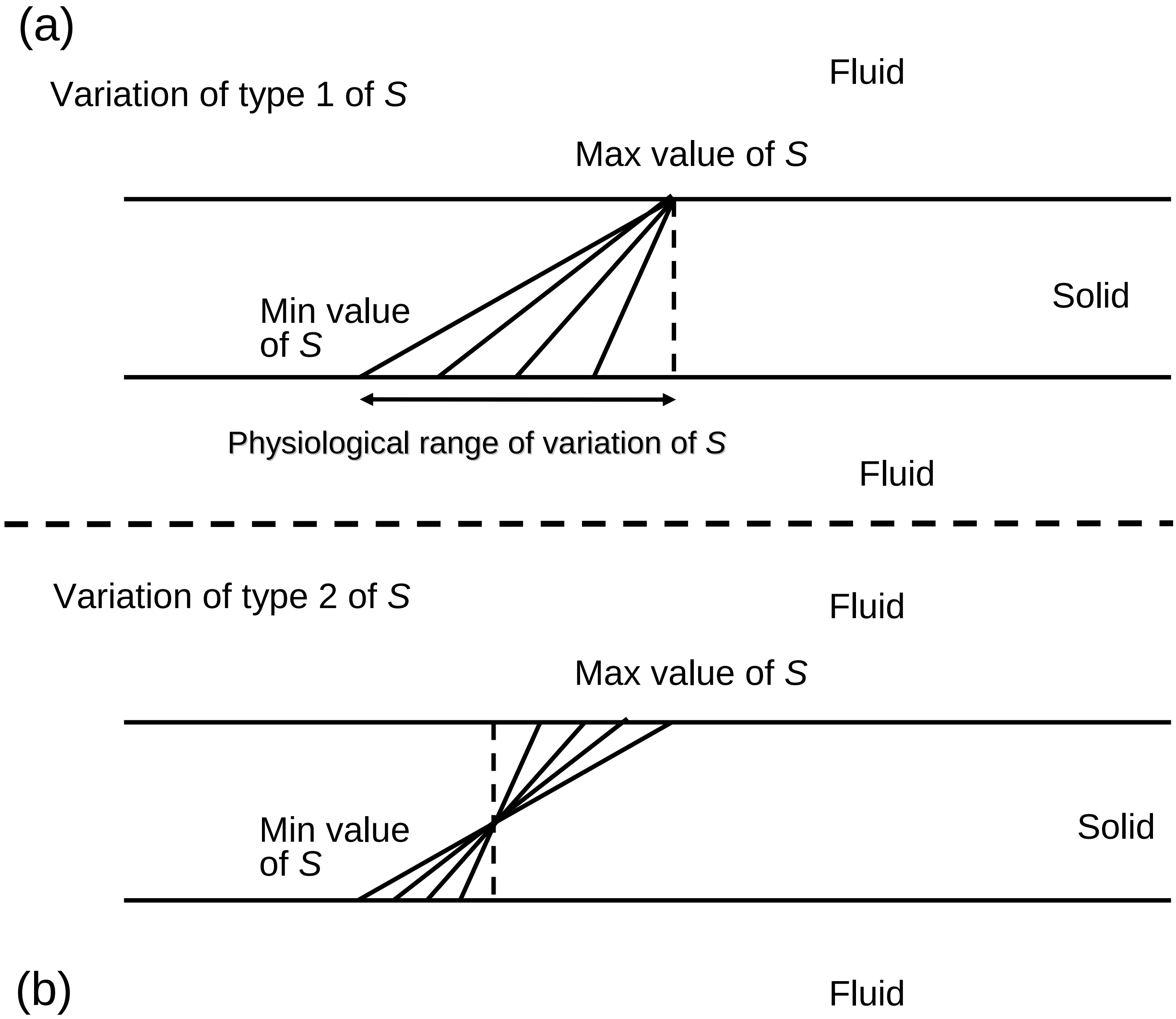
(a)
Variation of type 1 of S
Fluid
Max value of S
Solid
Min value
of S
Physiological range of variation of S
Fluid
Variation of type 2 of S
Fluid
Max value of S
Solid
Min value
of S
(b)
Fluid

Figure 3.

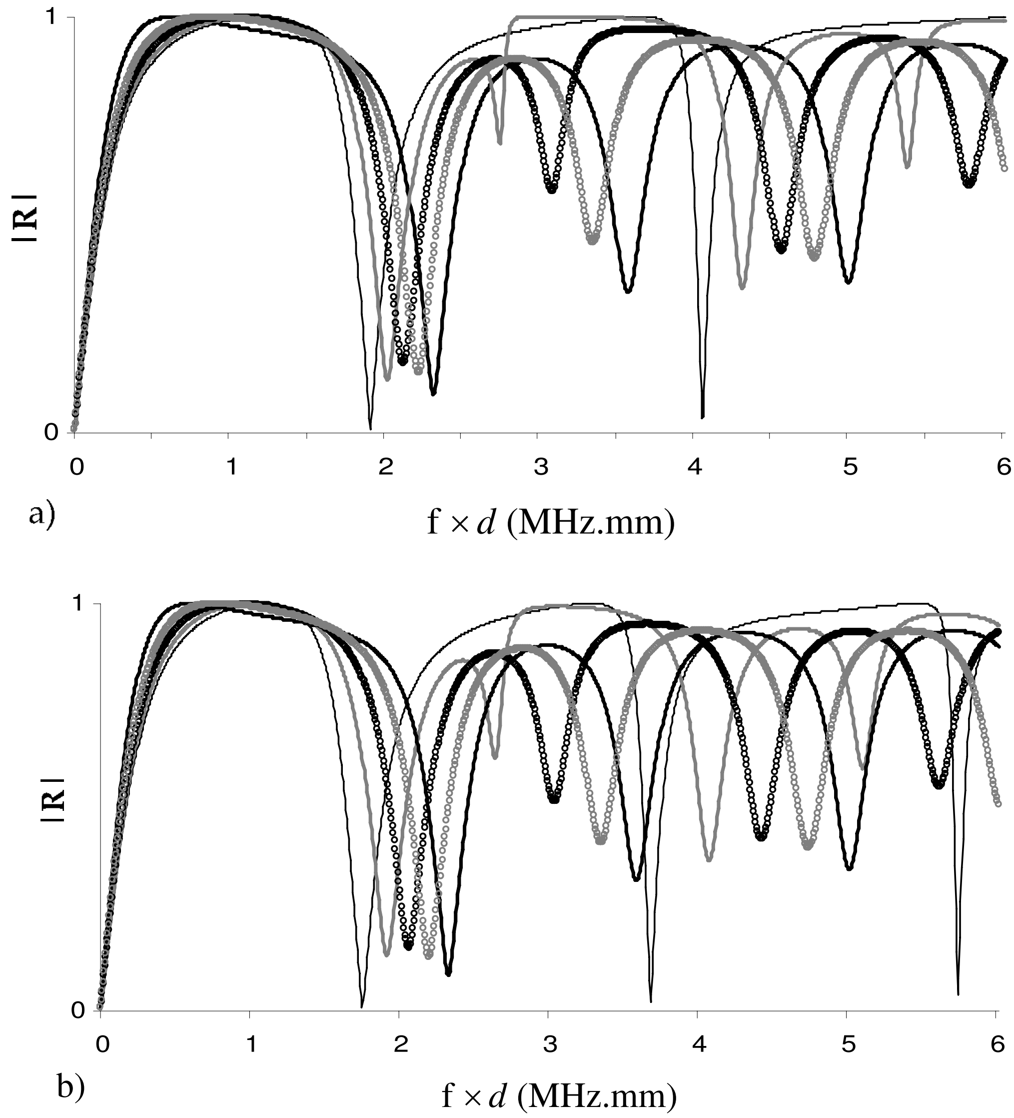

Figure 4.

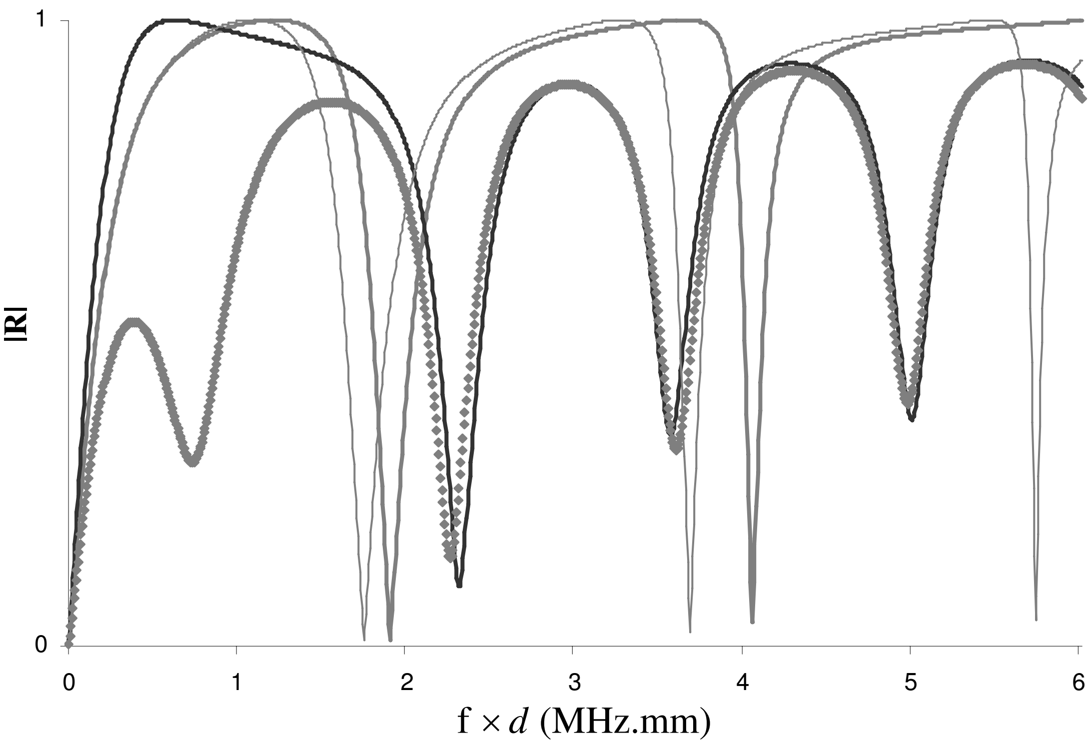

Figure 5.

LABORATOIRE
de MÉCANIQUE PHYSIQUE
UMR CNRS 7052 – B2OA

Salah NAILI, Professor

Créteil, January 14, 2009

Object: **Revised manuscript #08-05934**

Dear Professor McDaniel,

Thank you for the opportunity you gave us to revise our paper entitled “Propagation of elastic waves in a fluid-loaded anisotropic functionally graded waveguide: application to ultrasound characterization” by C. Baron and S. Naili. Please accept this enclosed revised version for reconsideration and publication in Journal of the Acoustical Society of America.

In response to the referees's suggestions and comments, we have revised our manuscript the best we could.

Principally, i) the advantages of the method are more clearly presented; ii) the section IV is devoted to the validation of the method; iii) the original results are described in the section V.

Hopefully, we did not miss any significant point. We hope that this corrected version provides an answer to each of your comments.

You will find with this letter for your convenience the point-by-point responses to the referees' comments.

Correspondence should be addressed to Professor Salah Naili.

Sincerely yours,

Salah NAILI,
for the authors.

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Salah NAILI, Professor

Créteil, January 14, 2009

Response to the referee 1 on the manuscript MS# 08-05934 presented by Baron and Naili
via
James McDaniel, Associated Editor
Journal of the Acoustical Society of America

Dear Referee,

Thank you for the opportunity you gave us to revise our paper entitled "Propagation of elastic waves in a fluid-loaded anisotropic functionally graded waveguide: application to ultrasound characterization" by C. Baron and S. Naili. Please accept this enclosed revised version for reconsideration and publication in Journal of the Acoustical Society of America.

We thank you for your helpful remarks which have allowed to improve the manuscript.

In response to the suggestions and comments of both referees, we have revised our manuscript the best we could.

In what follows you will find a point-by-point response to your comments, referred to their number (your comments are reproduced in small print for your convenience).

Reviewer #1 (Good Scientific Quality):

1. This paper proposes an analytical method to model waveguides with varying elastic properties along the transverse dimension. The method is original and based on previous works of the author. My main feeling after reading this manuscript is that, regardless of the method, the manuscript seems unaccomplished in the sense that the advantages of the method are not clearly presented: As an example in the conclusion it is said 'low-frequency asymptotics are naturally assessed evaluating only two or three terms'. It has not been shown at all in the paper (only a 70 terms expansion is considered). The following of the conclusion is around the numerical efficiency of the method and once again, it is not shown in the paper. If it is true, the author should present a quantitative result about the comparison of this method with classical ones in terms of computational time.

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This question has been treated in details in a previous paper (A.L. Shuvalov, O. Poncelet, M. Deschamps, C. Baron, Long-wavelength dispersion of acoustic waves in transversely inhomogeneous anisotropic plates, Wave Motion 42 (2005), pp. 367-382). We have added some sentences in section IV to summarize the main results of this paper.

2. Only 1 original result is presented (subsection IV.B.2), the other ones are comparison with the literature.

   From the section IV, the paper has been reorganized. In the revised version, the section IV is devoted to the validation of the method while as the section V presents the applications of the method as original results. We focus on the effect of a spatial gradient of material properties (mass density and stiffness coefficients) of cortical bone on its ultrasonic response obtained with an axial transmission device in the *in vivo* configuration.

3. Section IV.C does not provide any significant results; it is only general considerations. If the authors are true, they have to show the relevancy of their method in this context instead of conjecturing it. If the authors have significant results concerning this aspect; I won't accept that they publish two papers: one with the model and another one with the results.

   See the item 2.

Reviewer #1 (Tables/Figures Adequate):

I think that figures must be enhanced. Figure 1 is not necessary. The comparison of figure 2a. and 2.b is not easy.

Accordingly with the referee's suggestion, the figure 1 has been deleted. The figures have been revised. As suggested by the referee, the figure 1 has been deleted. The curves of figures 2.a and 2.b are in perfect agreement. As a consequence, only one series of dispersion curves would be visible if both were superimpose. That is the reason why, the authors decide to keep the former presentation and add a comment in the text to explain their choice.

Reviewer #1 (Concise):

Section IV.C is unnecessary.

The section IV has been reorganized. See item 2 above.

Reviewer #1 (Remarks):

Thank you for these helpful remarks. All remarks done by the referee have been incorporated in the revised version.

- Page 3: "The accuracy of the numerical .... perfectly managed" Give adequate references.

  The change has been made accordingly to the referee's suggestion.

- Page 5 section III: Avoid italics for the origin O of the frame.

  The change has been made accordingly to the referee's suggestion.

- Page 5 section III: the structure is "A" two dimensional one.

  The change has been made accordingly to the referee's suggestion.

- Page 6 last but one line: I suggest that it is better to express the ki as function of the cos and sine than the cos and sine as function of the ki.

  The change has been made accordingly to the referee's suggestion.

- Page 7. first line: put a hat on R and T to be coherent with the following of the paper (or avoid the hat in the following).

  The variables with a hat correspond to variables in the Fourier domain, whereas the variables without a hat represent physical quantities (space-time domain). To clarify this point, we added the parameters from which the $R, T, \hat{R}$ and $\hat{T}$ depend on.

- Page 8, line 4. normal stress instead of normal stresses.

  The change has been made accordingly to the referee's suggestion.

- Page 9 equation 13 second and third line of the matrix it is better to write s1c13/c33 to avoid ambiguities.

  The change has been made accordingly to the referee's suggestion.

- Page 10 ref on Peano series.

  The change has been made accordingly to the referee's suggestion.

- Page 10 near equation (17). The condensation of the system from 4 to 2 lines should be better explained (non triviality of eq. (16) and a sentence on sigma13).

  The change has been made accordingly to the referee's suggestion.

- Page 10 eq. 18 a prime on a second eta vector would be appreciated to avoid ambiguity.

  The change has been made accordingly to the referee's suggestion.

- Page 11 subsection d. there is a > at first line.

  The change has been made accordingly to the referee's suggestion.

- Page 12 explicit the choice of the number of terms in the expansion and in the choice of the number of integration points.

  We added the following sentence: “These choices ensure the convergence of the solution and the accuracy of the results for a reasonable computation time (never exceeding few minutes on the desktop computer)”.

- page 12 A. second paragraph first line: few differenceS.

  The change has been made accordingly to the referee’s suggestion.

- page 12 A. last but one line: develloped.

  The change has been made accordingly to the referee’s suggestion.

- page 13. properties of water are explained page 12.

  The change has been made accordingly to the referee’s suggestion.

Hopefully, no significant points seem to have been ignored, so far as we know. We hope that this corrected version provides an answer at each of your comments.

Yours sincerely,

Salah NAILI,
for the authors.

UNIVERSITÉ PARIS 12 VAL de MARNE
FACULTÉ DES SCIENCES ET TECHNOLOGIE

LABORATOIRE
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Salah NAILI, Professor

Créteil, January 14, 2009

Response to the referee 2 on the manuscript MS# 08-05934 presented by Baron and Naili
via
James McDaniel, Associated Editor
Journal of the Acoustical Society of America

Dear Referee,

Thank you for the opportunity you gave us to revise our paper entitled "Propagation of elastic waves in a fluid-loaded anisotropic functionally graded waveguide: application to ultrasound characterization" by C. Baron and S. Naili. Please accept this enclosed revised version for reconsideration and publication in Journal of the Acoustical Society of America.

We thank you for your helpful remarks which have allowed to improve the manuscript.

In response to the referees's suggestions and comments, we have revised our manuscript the best we could.

In what follows you will find a point-by-point response to your comments, referred to their number (your comments are reproduced in small print for your convenience).

Reviewer #2 (Good Scientific Quality):

Continuous variations have been treated before, at least by Willis as a model for very rough surfaces about 15 years ago. Anyway, the idea of integrating the Stroh form of the equations is a straightforward one, and I'd be surprised if it hadn't been treated before. On the other hand, when evaluating the integrals numerically, these authors are effectively discretizing their properties through the thickness of the material. (They are using Simpson's rule, which amounts to replacing the exact property variation with a piecewise quadratic approximation.) Certainly, however, considering continuous property variations is not the norm in the field.

One reason that piecewise constant variations continues to be the norm in the field, however, is the problem of stability which is not addressed by these authors. At high (High in this case means high enough that there are strongly evanescent wave modes in the field.) frequencies, the propagator matrix formulation is known to lead to numerical instabilities, and even a formally exact solution turns out to be practically worthless. In that regime, something needs to be done to address the

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numerical instability. The usual remedies require piecewise uniform material properties. At lower frequencies, nearly any old thing will work ok.

We agree with the referee's comment. There is still a stability problem at high frequencies, but not more than in more classical methods derived from Thomson Haskell. One of the advantages of the method presented in this paper is to provide an analytical exact solution, calculated approximately and so to be able to estimate the accuracy of the results (the errors come from the truncation of the series and from the numerical evaluation of the integrals), which cannot be done for the Thomson-Haskell method which provides an exact solution to an approximated problem.

Moreover, to evaluate the matricant on a greater distance, it would be necessary to increase the truncation order, which would extend the computation time and induce numerical instabilities (this second point is much more important), due to the excessive dimensions of the multiple integrals. Here, it is better to limit the spatial extent of the computation by breaking the study interval into smaller segments, then calculate the matricants for each of these subintervals, and finally use the composition formula:

$$\mathbf{M}(d,0) = \mathbf{M}(d,d_1)\mathbf{M}(d_1,0).$$

This approach yields a more accurate evaluation of the global matricant on the whole study interval. Thus, for each specific matricant, the truncation order remains reasonable, ensuring the stability and accuracy of the numerical computation. But, in our sense it is not the key point in this paper.

Reviewer #2 (Good Scientific Quality):

In summary, this approach is reasonable. It is novel, but not groundbreaking. It suffers from stability limitations that are not considered or described by the authors. For the primary application motivating this work, NDE of bone, the approach is reasonable.

Therefore, I'd recommend a slight reshaping of the manuscript, placing more emphasis on the intended application, and less emphasis on the "method" as a general purpose method. I'd suggest formulating the manuscript as "Here's the application we want to consider – this is the approach we're choosing to solve this problem."

We agree with the referee's comment. From the section IV, the paper has been reorganized. In the revised version, the section IV is devoted to the validation of the method while as the section V presents the applications of the method as original results. In the sub-section V.B, we focus on the effect of a spatial gradient of material properties (mass density and stiffness coefficients) of cortical bone on its ultrasonic response obtained with an axial transmission device in the *in vivo* configuration.

Hopefully, no significant points seem to have been ignored, so far as we know. We hope that this corrected version provides an answer at each of your comments.

Yours sincerely,

Salah NAILI,

for the authors.