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On the nonlinear theory of hydrodynamic instability in flames

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Résumé. — Une équation d'évolution décrivant la dynamique d'un front de flamme de prémélange soumis à l'instabilité de Darrieus-Landau est obtenue dans l'approximation d'une faible dilatation du gaz. L'équation obtenue au deuxième ordre ne diffère de celle obtenue au premier ordre que par une modification des coefficients. Dans la deuxième partie du papier on présente un modèle de flamme dans lequel les effets de la vorticité de l'écoulement des gaz brûlés sont négligés.

Abstract. — In the context of the weak thermal-expansion approximation, we derive an equation describing flame front dynamics under conditions of Darrieus-Landau instability. We show that the second-order theory leads to an evolution equation that differs from that of the first-order theory only in its coefficients. We also discuss a hydrodynamic flame model based on the equations for purely irrotational flow of the gas.

1. Introduction.

The dynamics of a perturbed plane flame front $x = \varphi(y, t)$ under conditions of thermal-expansion-induced instability is described by a nonlinear equation

$$\varphi_t + \frac{1}{2}\varphi_y^2 = \frac{\gamma}{2} I\{\varphi\} \tag{1.1}$$

where

$$I\{\varphi\} = \frac{1}{2\pi}\iint |k|\, \varphi(y', t)\, \mathrm{e}^{ik(y-y')}\, \mathrm{d}y'\, \mathrm{d}k$$

corresponding to the principal term of the asymptotic expansion in terms of the thermal-expansion parameter γ, with the latter assumed to be small [1]

$$\gamma = \frac{\rho_- - \rho_+}{\rho_-}. \tag{1.2}$$

Here $\rho_+ = 1$, $\rho_- = (1-\gamma)^{-1}$ are the densities of the burnt and cold gases, respectively. The flame propagation velocity relative to the burnt gas is assumed to be constant and equal to unity. In this approximation, the dynamics of the perturbed front is not sensitive to the vertical (y) component of the gas velocity vector. The perturbed flow of gas is seen as if it were unidirectional (1).

However, as recently noted by Clavin [2], allowance for refraction of the gas streamlines (Fig. 1) may result in the addition of a new term quadratic in φ to the equation for the dynamics of the front.

Indeed, the condition that the flame propagation velocity relative to the burnt gas be constant is expressed by

$$\mathbf{u}_+ \cdot \mathbf{n} - D = 1 \quad \text{at}\; x = \varphi(y, t) \tag{1.3}$$

where

$$\mathbf{n} = \left(1, -\varphi_y\right) \Big/ \sqrt{1+\varphi_y^2},$$
$$D = \varphi_t \Big/ \sqrt{1+\varphi_y^2},$$
$$\mathbf{u}_+ = \left(u_+, v_+\right).$$

(1) However, the tangential component of the gas velocity relative to the front may differ from zero. Thus the fact that the flow is unidirectional does not mean that a curved flame cannot exhibit a stretching effect.

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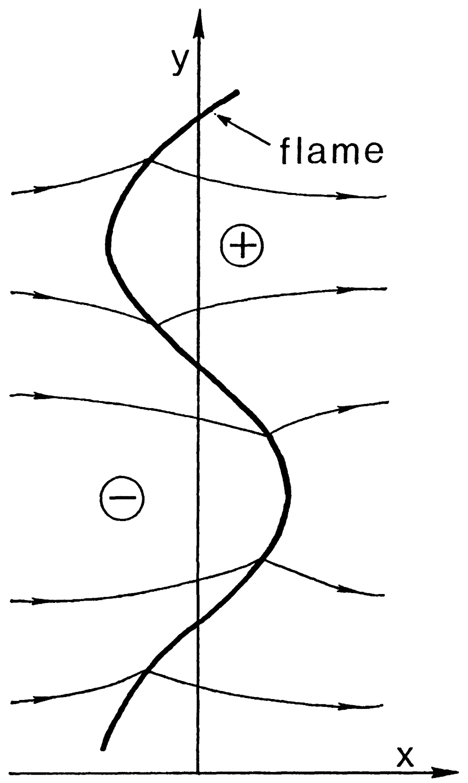

Fig. 1. — Diagram illustrating gas flow in a curved flame.

Written explicitly, this condition becomes

$$u_+(\varphi, y, t) - \varphi_y v_+(\varphi, y, t) - \varphi_t = = \sqrt{1+\varphi_y^2}. \quad (1.4)$$

Retaining only quadratic nonlinearities, we deduce that for small φ

$$u_+(0, y, t) + \varphi \frac{\partial u_+(0, y, t)}{\partial x} - - \varphi_y v_+(0, y, t) - \varphi_t = 1 + \frac{1}{2}\varphi_y^2. \quad (1.5)$$

Hence, using the continuity equation, we obtain

$$\varphi_t + \frac{1}{2}\varphi_y^2 = u_+ - 1 - \left(\varphi v_+\right)_y, \quad x = 0. \quad (1.6)$$

If $\gamma \ll 1$, it is known [1] that

$$v_+(0, y, t) = -\frac{1}{2}\gamma\varphi_y + O\left(\gamma^3\right)$$
$$u_+(0, y, t) = 1 + \frac{\gamma}{2} I\{\varphi\} + O\left(\gamma^3\right) \quad (1.7)$$
$$\left(\varphi \sim \gamma, \quad t \sim \gamma^{-1}\right).$$

Thus, the nonlinear convective term

$$\left(\varphi v_+\right)_y = -\frac{1}{2}\gamma\left(\varphi\varphi_y\right)_y + O\left(\gamma^4\right) \sim \gamma^3 \quad (1.8)$$

is small in comparison with the other terms of (1.6), and exerts no influence on flame front dynamics in the first approximation. However, in a more accurate description the nonlinear term (1.8) may lead to modification of the flame front equation (1.1). In this paper we attempt to construct an evolution equation incorporating the nonlinear term (1.8). As in the derivation of equation (1.1), γ is chosen as the expansion parameter.

2. Fundamental equations.

Within the limits of the Darrieus-Landau hydrodynamic theory, a flame is regarded as a surface of density discontinuity, propagating at constant velocity relative to the gas ; the gas itself is assumed to be incompressible an non-viscous. Thus, the gas flow in the regions of unburnt (−) and burnt (+) gas is described by the following system of Euler equations :

$$\frac{\partial \mathbf{u}_\pm}{\partial t} + \left(\mathbf{u}_\pm . \nabla\right)\mathbf{u}_\pm = \frac{1}{\rho_\pm}\nabla p_\pm, \quad \nabla . \mathbf{u}_\pm = 0 \quad (2.1)$$
$$\mathbf{u}_\pm = \left(u_\pm, v_\pm\right), \quad \rho_+ = 1, \quad \rho_- = (1-\gamma)^{-1}.$$

On the flame front $(x = \varphi(y, t))$ we have condition (1.3) and also conditions representing the continuity of both

i) mass flow :

$$\rho_+\left(\mathbf{u}_+ . \mathbf{n} - D\right) = \rho_-\left(\mathbf{u}_- . \mathbf{n} - D\right) \quad (2.2)$$

and ii) momentum flow :

$$\rho_+\left(\mathbf{u}_+ - D\mathbf{n}\right)\left(\mathbf{u}_+ . \mathbf{n} - D\right) + p_+\mathbf{n} = = \rho_-\left(\mathbf{u}_- - D\mathbf{n}\right)\left(\mathbf{u}_- . \mathbf{n} - D\right) + p_-\mathbf{n}. \quad (2.3)$$

Using (1.3), we can rewrite the two conditions as follows :

$$\mathbf{u}_+ . \mathbf{n} = \gamma + \mathbf{u}_- . \mathbf{n} \quad (2.4)$$
$$\mathbf{u}_+ . \boldsymbol{\tau} = \mathbf{u}_- . \boldsymbol{\tau} \quad (\boldsymbol{\tau} . \mathbf{n} = 0) \quad (2.5)$$
$$p_+ = -\gamma + p_-. \quad (2.6)$$

The basic solution of problem (1.3), (2.1), (2.2), (2.3) corresponding to an unperturbed plane flame $(\varphi = 0)$ is

$$u_+^{(b)} = 1, \quad v_+^{(b)} = 0, \quad p_+^{(b)} = 1 \quad (x > 0)$$
$$u_-^{(b)} = 1 - \gamma, \quad v_-^{(b)} = 0,$$
$$p_-^{(b)} = 1 + \gamma \quad (x < 0). \quad (2.7)$$

Since the model we are using does not include a typical length, the coordinates x, y may be assumed of the order of unity. We thus obtain the following estimates for the perturbed flame, on the assumption that $\gamma \ll 1$ [1]

$$\varphi \sim \gamma; \quad u_\pm - u_\pm^{(b)} \sim \gamma^2; \quad v_\pm - v_\pm^{(b)} \sim \gamma^2; \quad p_\pm - p_\pm^{(b)} \sim \gamma^2; \quad x, y \sim 1; \quad t \sim \gamma^{-1}. \quad (2.8)$$

Accordingly, we introduce scaled variables :

$$u_+ = 1 + \gamma^2 U_+ , \quad v_+ = \gamma^2 V_+ , \quad p_+ = 1 + \gamma^2 P_+ ,$$
$$u_- = 1 - \gamma + \gamma^2 U_- , \quad v_- = \gamma^2 V_- ,$$
$$p_- = 1 + \gamma + \gamma^2 P_- , \quad \varphi = \gamma \Phi , \quad \gamma t = \tau . \tag{2.9}$$

For the sequel, it is convenient to transform to curvilinear coordinates ξ, η attached to the perturbed flame front :

$$\xi = x - \gamma \Phi (y, \tau) , \quad \eta = y . \tag{2.10}$$

In terms of the new variables and parameters, equations (2.1) become

$$\frac{\partial U_\pm}{\partial \xi} + \frac{\partial P_\pm}{\partial \xi} = - \gamma \frac{\partial U_\pm}{\partial \tau} + \mathrm{O}(\gamma^2) , \tag{2.11}$$

$$\frac{\partial V_\pm}{\partial \xi} + \frac{\partial P_\pm}{\partial \eta} = - \gamma \frac{\partial V_\pm}{\partial \tau} + \gamma \Phi_\eta \frac{\partial P_\pm}{\partial \xi} + \mathrm{O}(\gamma^2) , \tag{2.12}$$

$$\frac{\partial U_\pm}{\partial \xi} + \frac{\partial V_\pm}{\partial \eta} = \gamma \Phi_\eta \frac{\partial V_\pm}{\partial \xi} . \tag{2.13}$$

Conditions (2.4), (2.5), (2.6) and (1.3) become

$$U_+ - U_- = - \frac{1}{2} \gamma \Phi_\eta^2 + \mathrm{O}(\gamma^2) , \tag{2.14}$$
$$V_+ - V_- = - \Phi_\eta + \mathrm{O}(\gamma^2) , \tag{2.15}$$
$$P_+ - P_- = 0 , \tag{2.16}$$
$$\Phi_\tau + \frac{1}{2} \Phi_\eta^2 = U_+ - \gamma V_+ \Phi_\eta + \mathrm{O}(\gamma^2) . \tag{2.17}$$

To solve problem (2.11)-(2.17), we assume the solution expressed as an asymptotic expansion :

$$\begin{aligned} \Phi &= \Phi^{(0)} + \gamma \Phi^{(0)} + \cdots \\ U_\pm &= U_\pm^{(0)} + \gamma U_\pm^{(1)} + \cdots \\ V_\pm &= V_\pm^{(0)} + \gamma V_\pm^{(1)} + \cdots \\ P_\pm &= P_\pm^{(0)} + \gamma P_\pm^{(1)} + \cdots \end{aligned} \tag{2.18}$$

3. First approximation.

In the first approximation, the equations and conditions of (2.11)-(2.17) are

$$\frac{\partial U_\pm^{(0)}}{\partial \xi} + \frac{\partial P_\pm^{(0)}}{\partial \xi} = 0 , \tag{3.1}$$
$$\frac{\partial V_\pm^{(0)}}{\partial \xi} + \frac{\partial P_\pm^{(0)}}{\partial \eta} = 0 , \tag{3.2}$$
$$\frac{\partial U_\pm^{(0)}}{\partial \xi} + \frac{\partial V_\pm^{(0)}}{\partial \eta} = 0 . \tag{3.3}$$
$$U_+^{(0)} - U_-^{(0)} = 0 , \tag{3.4}$$
$$V_+^{(0)} - V_-^{(0)} = - \Phi_\eta^{(0)} , \tag{3.5}$$
$$P_+^{(0)} - P_-^{(0)} = 0 , \tag{3.6}$$
$$\Phi_\tau^{(0)} + \frac{1}{2} \left(\Phi_\eta^{(0)} \right)^2 = U_+^{(0)} . \tag{3.7}$$

The solution of system (3.1)-(3.3) can be expressed as

$$\begin{aligned} U_\pm^{(0)} &= \frac{\partial \Pi_\pm^{(0)}}{\partial \xi} + R_\pm^{(0)} (\eta, \tau) , \\ V_\pm^{(0)} &= \frac{\partial \Pi_\pm^{(0)}}{\partial \eta} , \\ P_\pm^{(0)} &= - \frac{\partial \Pi_\pm^{(0)}}{\partial \xi} , \end{aligned} \tag{3.8}$$

where the potential $\Pi_\pm^{(0)}$ is a solution of the Laplace equation

$$\frac{\partial^2 \Pi_\pm^{(0)}}{\partial \xi^2} + \frac{\partial^2 \Pi_\pm^{(0)}}{\partial \eta^2} = 0 . \tag{3.9}$$

Hence

$$\Pi_\pm^{(0)} = \frac{1}{2\pi} \iint Q_\pm^{(0)} \times (\eta', \tau)^{ik(\eta - \eta') \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' . \tag{3.10}$$

The function $R_\pm^{(0)}$ is related to the flow vorticity :

$$\frac{\partial U_\pm^{(0)}}{\partial \eta} - \frac{\partial V_\pm^{(0)}}{\partial \xi} = \frac{\partial R_\pm^{(0)}}{\partial \eta} . \tag{3.11}$$

The flow ahead of the flame front is assumed to be irrotational :

$$R_-^{(0)} = 0 . \tag{3.12}$$

Inserting (3.8), (3.12) into equations (3.4)-(3.6), we obtain

$$R_+^{(0)} = 0, \quad Q_+^{(0)} = - \frac{1}{2} \Phi^{(0)} , \quad Q_-^{(0)} = \frac{1}{2} \Phi^{(0)} . \tag{3.13}$$

Thus,

$$U_\pm^{(0)} = \pm \frac{1}{4\pi} \iint |k| \, \Phi^{(0)} (\eta', \tau) \, \mathrm{e}^{ik(\eta - \eta') \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' \tag{3.14}$$

$$V_\pm^{(0)} = \mp \frac{1}{4\pi} \iint \Phi_\eta^{(0)} (\eta', \tau) \, \mathrm{e}^{ik(\eta - \eta') \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' \tag{3.15}$$

$$P_\pm^{(0)} = \mp \frac{1}{4\pi} \iint |k| \, \Phi^{(0)} (\eta', \tau) \, \mathrm{e}^{ik(\eta - \eta') \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' \tag{3.16}$$

Inserting (3.14) into condition (3.7), we finally have

$$\Phi_{\tau}^{(0)} + \frac{1}{2}\left(\Phi_{y}^{(0)}\right)^{2} = \frac{1}{2} I\left\{\Phi^{(0)}\right\}. \quad (3.17)$$

Returning to the original variables, we obtain equation (1.1).

We reiterate that, in the first approximation, the dynamic condition (2.17) is not sensitive to the vertical (y) velocity component $V_{+}^{(0)}$ (see (3.7)).

4. Second approximation.

The equations and conditions for the second approximation are :

$$\frac{\partial U_{\pm}^{(1)}}{\partial \xi} + \frac{\partial P_{\pm}^{(1)}}{\partial \xi} = -\frac{\partial U_{\pm}^{(0)}}{\partial \tau}, \quad (4.1)$$

$$\frac{\partial V_{\pm}^{(1)}}{\partial \xi} + \frac{\partial P_{\pm}^{(1)}}{\partial \eta} = -\frac{\partial V_{\pm}^{(0)}}{\partial \tau} + \Phi_{\eta}^{(0)} \frac{\partial P_{\pm}^{(0)}}{\partial \xi}, \quad (4.2)$$

$$\frac{\partial U_{\pm}^{(1)}}{\partial \xi} + \frac{\partial V_{\pm}^{(1)}}{\partial \eta} = \Phi_{\eta}^{(0)} \frac{\partial V_{\pm}^{(0)}}{\partial \xi}, \quad (4.3)$$

$$U_{+}^{(1)} - U_{-}^{(1)} = -\frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^{2}, \quad (4.4)$$

$$V_{+}^{(1)} - V_{-}^{(1)} = -\Phi_{\eta}^{(1)}, \quad (4.5)$$

$$P_{+}^{(1)} - P_{-}^{(1)} = 0, \quad (4.6)$$

$$\Phi_{\tau}^{(1)} + \Phi_{\eta}^{(0)} \Phi_{\eta}^{(1)} = U_{+}^{(1)} - \Phi_{\eta}^{(0)} V_{+}^{(0)}. \quad (4.7)$$

The solution of system (4.1)-(4.3) can be expressed as

$$U_{\pm}^{(1)} = \frac{\partial \Pi_{\pm}^{(1)}}{\partial \xi} + R_{\pm}^{(1)}(\eta, \tau), \quad (4.8)$$

$$V_{\pm}^{(1)} = \frac{\partial \Pi_{\pm}^{(1)}}{\partial \eta} - \Phi_{\eta}^{(0)} \frac{\partial \Pi_{\pm}^{(0)}}{\partial \xi}, \quad (4.9)$$

$$P_{\pm}^{(1)} = -\frac{\partial \Pi_{\pm}^{(1)}}{\partial \xi} - \frac{\partial \Pi_{\pm}^{(0)}}{\partial \tau}, \quad (4.10)$$

where the potential $\Pi_{\pm}^{(1)}$ satisfies the equation

$$\frac{\partial^{2} \Pi_{\pm}^{(1)}}{\partial \xi^{2}} + \frac{\partial^{2} \Pi_{\pm}^{(1)}}{\partial \eta^{2}} = \Phi_{\eta\eta}^{(0)} \frac{\partial \Pi_{\pm}^{(0)}}{\partial \xi} + 2\, \Phi_{\eta}^{(0)} \frac{\partial^{2} \Pi_{\pm}^{(0)}}{\partial \xi\, \partial \eta}. \quad (4.11)$$

Hence

$$\Pi_{\pm}^{(1)} = \Phi^{(0)} \frac{\partial \Pi_{\pm}^{(0)}}{\partial \xi} + \frac{1}{2\pi} \times \iint Q_{\pm}^{(1)}(\eta', \tau)\, \mathrm{e}^{ik(\eta-\eta') \mp |k| \xi}\, \mathrm{d}k\, \mathrm{d}\eta'. \quad (4.12)$$

By assumption, ahead of the flame front we have

$$R_{-}^{(1)} = 0. \quad (4.13)$$

Inserting (4.8), (4.9), (4.10) and (4.13) into conditions (4.4), (4.5) and (4.6), we obtain after some manipulation

$$R_{+}^{(1)} = -\frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^{2} - \Phi_{\tau}^{(0)} \quad (4.14)$$

$$I\left\{Q_{\pm}^{(1)}\right\} = \mp \frac{1}{2} I\left\{\Phi^{(1)}\right\} + \frac{1}{2} \Phi^{(0)} \Phi_{\eta\eta}^{(0)} - \frac{1}{2} \Phi_{\tau}^{(0)}. \quad (4.15)$$

We now consider the dynamic condition (4.7). From (4.12) and (4.15) we have

$$\frac{\partial H_{+}^{(1)}(0, \eta, \tau)}{\partial \xi} = \frac{1}{2} \Phi^{(0)} \Phi_{\eta\eta}^{(0)} - I\left\{Q_{+}^{(1)}\right\} = \frac{1}{2} I\left\{\Phi^{(1)}\right\} + \frac{1}{2} \Phi_{\tau}^{(0)}. \quad (4.16)$$

Thus,

$$U_{+}^{(1)}(0, \eta, \tau) = \frac{\partial \Pi_{+}^{(1)}(0, \eta, \tau)}{\partial \xi} + R_{+}^{(1)}(\eta, \tau) = \frac{1}{2} I\left\{\Phi^{(1)}\right\} - \frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^{2} - \frac{1}{2} \Phi_{\tau}^{(0)} = \frac{1}{2} I\left\{\Phi^{(1)}\right\} - \frac{1}{4} I\left\{\Phi^{(0)}\right\} - \frac{1}{4}\left(\Phi_{\eta}^{(0)}\right)^{2}. \quad (4.17)$$

Here we have also used equation (3.17). By (3.15),

$$V_{+}^{(0)}(0, \eta, \tau)\, \Phi_{\eta}^{(0)} = -\frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^{2}. \quad (4.18)$$

Inserting (4.17), (4.18) into condition (4.7), we arrive at a closed equation for $\Phi^{(1)}$:

$$\Phi_{\tau}^{(1)} + \Phi_{\eta}^{(0)} \Phi_{\eta}^{(1)} = \frac{1}{2} I\left\{\Phi^{(1)}\right\} - \frac{1}{4} I\left\{\Phi^{(0)}\right\} + \frac{1}{4}\left(\Phi_{\eta}^{(0)}\right)^{2}. \quad (4.19)$$

Synthesizing equations (4.19) and (3.17), we obtain

$$\Phi_\tau + \frac{1}{2}\Phi_\eta^2 = \frac{1}{2}\left(1 - \frac{1}{2}\gamma\right) I\{\Phi\} + \frac{1}{4}\gamma\Phi_\eta^2 . \quad (4.20)$$

Hence, in terms of the original variables,

$$\varphi_\tau + \frac{1}{2}\left(1 - \frac{\gamma}{2}\right)\varphi_\eta^2 = \frac{\gamma}{2}\left(1 - \frac{\gamma}{2}\right) I\{\varphi\} . \quad (4.21)$$

Thus, it turns out that the nonlinear convective term (1.8) is *partially* neutralized by the nonlinear correction induced by the component u_+. As a result, equation (4.21) differs from equation (1.1) only in its coefficients. In other words, despite the more accurate description we have adopted, the qualitative picture of flame front dynamics remains unchanged. The flame front equation undergoes qualitative change only in the third order theory, where new quadratic terms stemming from the $(\mathbf{u} \cdot \nabla)\, \mathbf{u}$ term of the Euler equations should appear. In the second order theory the contribution of these terms is not taken into account.

Equation (4.21) yields the following dispersion relation corresponding to the linear stability problem:

$$\omega = \frac{1}{2}\gamma\left(1 - \frac{1}{2}\gamma\right) | k | \quad (\varphi \sim \exp(\omega t + ik\eta)) . \quad (4.22)$$

This result is in agreement with the known dispersion relation, obtained for arbitrary γ (Landau, 1944):

$$\omega = \frac{\gamma - 1 + \sqrt{1 - 2\gamma^2 + \gamma^3}}{2 - \gamma} | k | = \left[\frac{\gamma}{2}\left(1 - \frac{\gamma}{2}\right) + O(\gamma^3)\right] | k | . \quad (4.23)$$

5. Irrotational model for hydrodynamic flame instability.

In the first approximation with respect to γ, if the flow is irrotational ahead of the flame front $\left(R_-^{(0)} = 0\right)$, it will also be irrotational behind the front $\left(R_+^{(0)} = 0\right)$. Consequently, the production of vorticity, which is generally peculiar to a curved flame front [3], is not a decisive factor in hydrodynamic flame instability. Thus, for small γ, flame instability can be described correctly within the limits of a model based on the equations of purely irrotational flow:

$$\frac{\partial u_\pm}{\partial y} - \frac{\partial v_\pm}{\partial x} = 0$$
$$\frac{\partial u_\pm}{\partial x} + \frac{\partial v_\pm}{\partial y} = 0 \quad (5.1)$$

with the three conditions (1.3), (2.4) and (2.5) satisfied on the flame front.

In the first approximation (with respect to γ), such a model is asymptotically identical to that based on the Euler equations (2.1). The condition (2.6) for the pressure jump is then automatically satisfied. This is no longer the case in the second approximation.

Being less accurate, though easier to handle from the mathematical point of view, the « irrotational » model may prove extremely instructive for an understanding of many nonlinear phenomena in flame instabilities. For example, the model reveals interesting prospects for the description of finite-amplitude wrinkled flames using methods of complex analysis.

If condition (1.3) is replaced by Markstein's condition [4] relating the flame velocity to its curvature, the irrotational model can also describe the self-turbulence effect in flames [5].

In this paper we shall confine ourselves to deriving a dynamic equation for the flame front incorporating effects of second order in γ.

In terms of a potential $w_\pm$, problem (5.1), (2.4), (2.5), (1.3) may be written as follows:

$$\frac{\partial^2 w_\pm}{\partial x^2} + \frac{\partial^2 w_\pm}{\partial y^2} = 0 . \quad (5.2)$$

At $x = \varphi(y, t)$:

$$\frac{\partial w_+}{\partial n} = \gamma + \frac{\partial w_-}{\partial n} , \quad (5.3)$$

$$w_+ = w_- , \quad (5.4)$$

$$\frac{\partial w_+}{\partial n} - D = 1 . \quad (5.5)$$

The basic solution, corresponding to an unperturbed plane flame, is

$$w_+^{(b)} = x , \quad w_-^{(b)} = (1 - \gamma)\, x . \quad (5.6)$$

Proceeding as previously (Sect. 2), we introduce scaled variables $W_\pm$, Φ, τ:

$$w_+ = x + \gamma^2 W_+ , \quad w_- = (1 - \gamma)\, x + \gamma^2 W_-$$
$$\varphi = \gamma\Phi , \quad \tau = \gamma t \quad (5.7)$$

and transform to curvilinear coordinates ξ, η as in (2.10). In terms of the new variables and parameters, problem (5.2)-(5.5) becomes

$$\frac{\partial^2 W_\pm}{\partial \xi^2} + \frac{\partial^2 W_\pm}{\partial \eta^2} = \gamma\Phi_{\eta\eta}\frac{\partial W_\pm}{\partial \xi} + 2\gamma\Phi_\eta \frac{\partial^2 W_\pm}{\partial \xi\, \partial \eta} + O(\gamma^2) . \quad (5.8)$$

At $\xi = 0$:

$$\frac{\partial W_+}{\partial \xi} = \frac{\partial W_-}{\partial \xi} - \frac{1}{2}\gamma\Phi_\eta^2 + O(\gamma^2) , \quad (5.9)$$

$$W_+ = W_- - \Phi , \quad (5.10)$$

$$\Phi_\tau + \frac{1}{2}\Phi_\eta^2 = \frac{\partial W_+}{\partial \xi} - \gamma\Phi_\eta \frac{\partial W_+}{\partial \eta} + O(\gamma^2) . \quad (5.11)$$

Again, we expand the desired solution of problem (5.8)-(5.11) in asymptotic series:

$$\begin{aligned} \Phi &= \Phi^{(0)} + \gamma \Phi^{(1)} + \cdots \\ W_{\pm} &= W_{\pm}^{(0)} + \gamma W_{\pm}^{(1)} + \cdots \end{aligned} \tag{5.12}$$

In the first approximation, equations (5.8)-(5.11) give

$$\frac{\partial^2 W_{\pm}^{(0)}}{\partial \xi^2} + \frac{\partial^2 W_{\pm}^{(0)}}{\partial \eta^2} = 0, \tag{5.13}$$

$$\frac{\partial W_{+}^{(0)}}{\partial \xi} = \frac{\partial W_{-}^{(0)}}{\partial \xi}, \tag{5.14}$$

$$W_{+}^{(0)} = W_{-}^{(0)} - \Phi^{(0)}, \tag{5.15}$$

$$\Phi_{\tau}^{(0)} + \frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^2 = \frac{\partial W_{+}^{(0)}}{\partial \xi}. \tag{5.16}$$

Hence (see Sect. 2),

$$W_{\pm}^{(0)} = \mp \frac{1}{4\pi} \iint \Phi^{(0)}(\eta', \tau) \times \times \mathrm{e}^{ik(\eta' - \eta) \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' \tag{5.17}$$

$$\Phi_{\tau}^{(0)} + \frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^2 = \frac{1}{2} I \left\{ \Phi^{(0)} \right\}. \tag{5.18}$$

In the second approximation, the equations are

$$\frac{\partial^2 W_{\pm}^{(1)}}{\partial \xi^2} + \frac{\partial^2 W_{\pm}^{(1)}}{\partial \eta^2} = = \Phi_{\eta\eta}^{(0)} \frac{\partial W_{\pm}^{(0)}}{\partial \xi} + 2\, \Phi_{\eta}^{(0)} \frac{\partial^2 W_{\pm}^{(0)}}{\partial \xi \, \partial \eta}, \tag{5.19}$$

$$\frac{\partial W_{+}^{(1)}}{\partial \xi} = \frac{\partial W_{-}^{(1)}}{\partial \xi} - \frac{1}{2}\left(\Phi_{\eta}^{(0)}\right)^2, \tag{5.20}$$

$$W_{+}^{(1)} = W_{-}^{(1)} - \Phi^{(1)} \tag{5.21}$$

$$\Phi_{\tau}^{(1)} + \Phi_{\eta}^{(0)} \Phi_{\eta}^{(1)} = = \frac{\partial W_{+}^{(1)}}{\partial \xi} - \Phi_{\eta}^{(0)} \frac{\partial W_{+}^{(0)}}{\partial \eta}. \tag{5.22}$$

From (5.19)-(5.21) there follows

$$W_{\pm}^{(1)} = \Phi^{(0)} \frac{\partial W_{\pm}^{(0)}}{\partial \xi} + + \frac{1}{2\pi} \iint G_{\pm}^{(1)}(\eta', \tau)\, \mathrm{e}^{ik(\eta - \eta') \mp |k| \xi} \, \mathrm{d}k \, \mathrm{d}\eta' \tag{5.23}$$

$$I\left\{ G_{\pm}^{(1)} \right\} = \frac{1}{2} \Phi^{(0)} \Phi_{\eta\eta}^{(0)} + + \frac{1}{4}\left(\Phi_{\eta}^{(0)}\right)^2 \mp \frac{1}{2} I \left\{ \Phi^{(1)} \right\}. \tag{5.24}$$

Inserting these relations in (5.22), we obtain

$$\Phi_{\tau}^{(1)} + \Phi_{\eta}^{(0)} \Phi_{\eta}^{(1)} = = \frac{1}{4}\left(\Phi_{\eta}^{(0)}\right)^2 + \frac{1}{2} I \left\{ \Phi^{(1)} \right\}. \tag{5.25}$$

Combining (5.25) with equation (5.18), we obtain

$$\Phi_{\tau} + \frac{1}{2}\left(1 - \frac{\gamma}{2}\right) \Phi_{\eta}^2 = \frac{1}{2} I \left\{ \Phi \right\} \tag{5.26}$$

or

$$\varphi_t + \frac{1}{2}\left(1 - \frac{\gamma}{2}\right) \varphi_y^2 = \frac{1}{2} \gamma I \left\{ \varphi \right\}. \tag{5.27}$$

Thus, in comparison with the exact model (4.21), neglect of vorticity generation increases the perturbation growth rate. However, the nonlinear terms in the two equations are completely identical.

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