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Self-focusing of thin liquid jets

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The nonlinear evolution of an initially perturbed free surface perpendicularly accelerated, or of an initially flat free surface subject to a perturbed velocity profile, gives rise to the emergence of thin spikes of fluid. We are investigating the long-time evolution of a thin inviscid jet of this kind, subject or not to a body force acting in the direction of the jet itself. A fully nonlinear theory for the long-time evolution of the jet is given. In two dimensions, the curvature of the tip scales like t^3, where t is time, and the peak undergoes an overshoot in acceleration which evolves like t^{-5}. In three dimensions, the jet evolves towards an axisymmetric shape, and the curvature and the overshoot in acceleration obey asymptotic laws in t^2 and t^{-4}, respectively. The asymptotic self-similar shape of the spike is found to be a hyperbola in two dimensions, a hyperboloid in three dimensions. Scaling laws and self-similarity are confronted with two-dimensional computations of the Richtmyer–Meshkov instability.

Keywords: potential flow; self-similarity; thin jets

1. Introduction

Free-surface thin jets of inviscid fluids occur in many different areas of fluid dynamics (Eggers 1997), including surface waves (Longuet-Higgins 1976, 1983; Longuet-Higgins & Dommermuth 2001), asymmetric bubble collapse (Blake *et al.* 1997; Prosperetti 1997; Popinet & Zaleski 2002), bursting of bubbles at a free surface (Duchemin *et al.* 2002), drop impact onto a solid surface (Bartolo *et al.* 2006) or even ink-jet printing (Dijksman 1999). Fundamental instabilities involving jets have been widely studied, among which are the Rayleigh–Taylor (RT; Rayleigh 1900; Taylor 1950) and Richtmyer–Meshkov (RM) instabilities (Richtmyer 1960; Meshkov 1969). These two instabilities correspond to the situation where a fluid is accelerated towards another one, the boundary between the two fluids being initially perturbed. In the case of the RT instability, a heavy fluid is subject to a constant body force like gravity acting in the direction of a lighter fluid, whereas in the case of the RM instability, the perturbed interface is impulsively accelerated by a shock wave, a situation which can be well approximated by imposing an impulsive acceleration perpendicular to the interface (Carles & Popinet 2001). Special attention has been drawn recently to the limit where the lighter fluid does not play a role in the evolution of the surface (Clavin & Williams 2005; Duchemin *et al.* 2005; Antkowiak *et al.* 2007),

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i.e. when its density is much lower than the one of the heavier fluid. In particular, this assumption is relevant in the context of water impact problems (Cooker & Peregrine 1995) and in the context of inertial confinement fusion (ICF; Duchemin *et al.* 2005). These recent studies of hydrodynamic instabilities occurring on ICF targets focus on the long-time dynamics of the tiny spikes observed in the RT instability, in two dimensions. In this context, it is shown that the spikes undergo an overshoot in acceleration scaling like t^{-5} and that the peak's curvature evolves like t^3, where t is time.

In this paper, we concentrate on the special case of infinite density ratio (i.e. a free-surface flow), for which the Kelvin–Helmholtz instability cannot occur; we neglect the effect of surface tension and assume that the fluid is incompressible. The type of 'free' jets we are interested in corresponds to jets without an imposed mass flux or velocity profile; they are free-evolving jets subject to a time-dependent body force and arise in any of the cited fundamental free-surface instabilities.

A general nonlinear theory is given for the spike evolution at long time, in both two and three dimensions. The theory is valid for jets subject or not to a body force acting in the direction of the jet itself. In the two-dimensional case, we recover the same type of self-similarity and scaling laws as the one found by Clavin & Williams (2005) for the special case of the RT instability. In three dimensions, self-similarity and scaling laws around the spike differ from the two-dimensional ones: the curvature scales like t^2 and the overshoot in acceleration like t^{-4}. It is also shown that at long time, any three-dimensional jet will end up being axisymmetric. Finally, predicted scaling laws are confronted with a two-dimensional simplified version of the RM instability. A good agreement is obtained between the theoretical scaling laws and self-similarity exponents and the numerical results.

2. Theory

We consider a vertical inviscid thin jet subject to a time-dependent body force $\boldsymbol{g}(t)$ pointing either upwards or downwards. Instead of solving this specific problem, we consider the rigorously equivalent configuration in which there is no body force and the fluid far from the jet is subject to an acceleration $-\boldsymbol{g}(t)$ (cf. figure 1). The theory developed in this paper will be valid for any function $\boldsymbol{g}(t)$, including the Dirac function.

This flow is completely described by the definition of a velocity potential φ satisfying Laplace's equation

$$\Delta\varphi = 0, \tag{2.1}$$

where φ is a function of space and time coordinates, and the velocities of fluid particles along x-, y- and z-axes are given by $u=\varphi_x$, $v=\varphi_y$ and $w=\varphi_z$, respectively, where the subscript denotes differentiation. Since we are looking for a velocity potential valid in a region where $|x|, |y| \ll 1$, symmetric about the z-axis, a second-order expansion in x and y of this potential gives

$$\varphi(x, y, z, t) = \varphi_0(z, t) + x^2\varphi_1(z, t) + y^2\varphi_2(z, t). \tag{2.2}$$

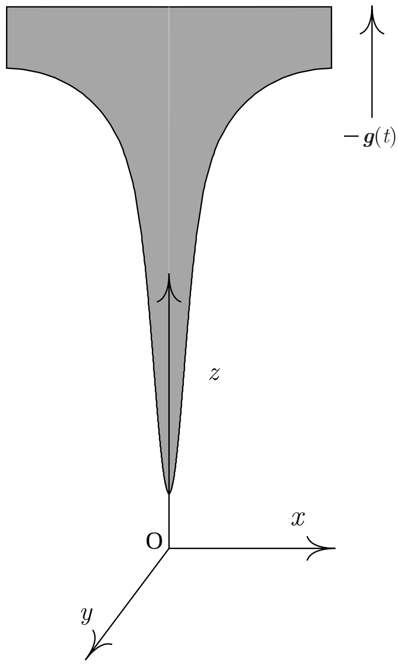

Figure 1. Galilean frame of reference according to which the flow is described. The fluid far from the jet moves away from this frame of reference with acceleration $-\boldsymbol{g}(t)$.

At this stage, we make the assumption that the flow is well described by a pure straining flow potential

$$\varphi = \frac{1}{2}\left(\frac{\dot{a}}{a}x^2 + \frac{\dot{b}}{b}y^2 + \frac{\dot{c}}{c}z^2\right), \tag{2.3}$$

where a, b and c are functions of time only. This notation is the one used by Longuet-Higgins (1983) and allows a natural way of finding the free-surface shape.

Although equation (2.3) seems to be a strong assumption, a physical argument can be given in favour of this choice. Indeed, asymptotically, we expect the trajectories of fluid particles in the jet to be Galilean, i.e. with velocity components (U, V, W) different for each particle and constant in time, and positions $X = Ut + X_0$, $Y = Vt + Y_0$ and $Z = Wt + Z_0$ where (X_0, Y_0, Z_0) is the initial position of the particle. In other words, we expect the pressure gradient in the fluid to be negligible, at large time, compared to inertia. The velocity components of fluid particles as a function of their current and initial positions read $U = (X - X_0)/t$, $V = (Y - Y_0)/t$ and $W = (Z - Z_0)/t$. Considering that at large time, the fluid particles are far from their initial position, the vertical velocity component can be approximated by $W \approx Z/t$. Such an asymptotic velocity corresponds to an Eulerian velocity potential quadratic in z. Using this fact in expansion (2.2) together with Laplace's equation, we find that φ_1 and φ_2 have to be functions of time only. This finally leads to a velocity potential of the form (2.3). As explained later, the linear behaviour in z of φ_z is validated thanks to the numerical study presented in this paper.

In order for φ to satisfy Laplace's equation, functions a, b and c must satisfy

$$abc = M, \tag{2.4}$$

where M is a constant. It is worth noting that functions a, b and c can be rescaled according to any constant, without altering the velocity potential φ. Only the value of M is changed. On the free surface, both the Bernoulli pressure $-2p = 2\varphi_t + \varphi_x^2 + \varphi_y^2 + \varphi_z^2 + f(t)$ and its Lagrangian derivative $(\partial_t + \varphi_x \partial_x + \varphi_y \partial_y + \varphi_z \partial_z)(-2p)$ are zero ($f(t)$ is an unknown function of time). Following Longuet-Higgins (1983), we replace the expression for φ in these equations and obtain:

$$\frac{\ddot{a}}{a} x^2 + \frac{\ddot{b}}{b} y^2 + \frac{\ddot{c}}{c} z^2 + f(t) = 0 \tag{2.5}$$

and

$$\frac{a\dddot{a} + \dot{a}\ddot{a}}{a^2} x^2 + \frac{b\dddot{b} + \dot{b}\ddot{b}}{b^2} y^2 + \frac{c\dddot{c} + \dot{c}\ddot{c}}{c^2} z^2 + \dot{f}(t) = 0. \tag{2.6}$$

In order for these equations to represent the same surface, the four terms in equations (2.5) and (2.6) have to be in proportion. Integrating these new relations according to time and rescaling a, b and c in order to remove the integration constants, we find

$$a\ddot{a} = b\ddot{b} = -c\ddot{c} = f(t). \tag{2.7}$$

Since we are interested in elongated jets, we consider only the two-sheeted hyperboloid case, which corresponds to a negative constant in front of $c\ddot{c}$ and positive constant in front of the other terms. Using the expression for the Bernoulli pressure on the surface and equation (2.7), we find the equation of the surface

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{h^2(x, y, t)}{c^2} = 1, \tag{2.8}$$

where $h(x, y, t)$ is the elevation of the free surface.

3. Three-dimensional flow

Asymptotically for long-time evolution, solutions to equations (2.4) and (2.7) are power series expansions of time $a \sim a_0 t^{\alpha}$, $b \sim b_0 t^{\beta}$ and $c \sim c_0 t^{\gamma}$ at leading order. The case $\alpha = 0$ or $\beta = 0$ corresponds to the two-dimensional flow (see §4). We exclude the case $\alpha = \beta = 0$ which implies $\gamma = 0$ from equation (2.4) and would correspond to a static configuration: it is impossible unless there is initially no fluid motion. Since the jet is thin, α and β are negative; then from the first equality in equation (2.7), we have at leading order

$$a_0^2 \alpha(\alpha - 1) t^{2\alpha-2} = b_0^2 \beta(\beta - 1) t^{2\beta-2}. \tag{3.1}$$

It follows that $\alpha = \beta$ and $a_0 = b_0$, which means that the flow and the free surface are asymptotically axisymmetric. From equation (2.4), we have $\alpha + \beta = 2\alpha = -\gamma$. Equation (2.7) leads to

$$a_0^2 \alpha(\alpha - 1) t^{2\alpha-2} = -2c_0^2 \alpha(2\alpha + 1) t^{-4\alpha-2}. \tag{3.2}$$

Then, $\alpha=-1/2$ and $\gamma=1$ is the only solution, and $a_0^2 c_0 = M$. Indeed, time exponents in equation (3.2) are unbalanced unless $\alpha=0$. Since we look for $\alpha<0$, the prefactors have to be then identically zero, and the balance is ensured at the next order in the expansion. At leading order, $a(t)=a_0 t^{-1/2}$ and $c(t)=c_0 t$. The leading-order term in $a\ddot{a}$ is $3a_0^2 t^{-3}/4 = 3Mt^{-3}/4c_0$, such that the next term in c is $c_1/t^2 = -M/8c_0^2 t^2$,

$$c(t) = c_0 t - \frac{M}{8c_0^2 t^2} + o(t^{-2}). \tag{3.3}$$

At this stage and for simplicity, we can choose $c_0=1$, which is equivalent to rescaling a, b and c according to c_0 and change the constant M. From the equation of the free surface, we extract the curvature at the tip

$$\kappa_0 = 2\partial_{xx} h(0,0,t) = 2\frac{c}{a^2} = 2\frac{c^2}{M} \sim 2\frac{t^2}{M} \tag{3.4}$$

and the vertical position, velocity and acceleration of the tip in the Galilean frame of reference, respectively,

$$z_t = t - \frac{M}{8t^2} + o(t^{-2}), \tag{3.5}$$

$$v_t = \partial_t h(0,0,t) = \dot{c} = 1 + \frac{M}{4t^3} + o(t^{-3}) \tag{3.6}$$

and

$$a_t = \partial_{tt} h(0,0,t) = \ddot{c} = -\frac{3M}{4t^4} + o(t^{-4}). \tag{3.7}$$

Finally, coming back to a frame of reference R', where the fluid is at rest at infinity, the acceleration of the tip reads

$$a_t\Big|_{R'} = g(t) - \frac{3M}{4t^4} + o(t^{-4}). \tag{3.8}$$

Therefore, knowing $1/M$ from the long-time evolution of the tip's curvature, we have direct access to the coefficient in front of the tip's overshoot in acceleration $3M/4t^4$. The asymptotic self-similar equation of the surface reads

$$-\frac{X^2+Y^2}{M^2} + Z^2 = 1, \tag{3.9}$$

where $X=xt^{1/2}$, $Y=yt^{1/2}$ and $Z=z/t$. We have shown that the curvature of the tip scales like t^2 and that the overshoot in acceleration undergone by the spike scales like t^{-4}.

4. Two-dimensional case

The two-dimensional case corresponds to the surface equation

$$-\frac{x^2}{a^2} + \frac{h^2(x,t)}{c^2} = 1, \tag{4.1}$$

where $h(x, t)$ is the elevation of the free surface and a and c satisfy

$$ac = M \quad \text{and} \quad a\ddot{a} = -c\ddot{c}. \tag{4.2}$$

As in the general case, functions a and c are power series expansions of time; then, from equations (4.2), it follows that the leading-order exponent of either a or c has to be equal to 1. Therefore, remembering that we are looking for thin jets, the leading-order exponent of a is -1 and c is 1. Using the second equation in (4.2), we deduce the next order coefficient and exponent in the expansion for c,

$$c(t) = c_0 t - \frac{M^2}{6c_0^3 t^3} + o(t^{-3}). \tag{4.3}$$

Choosing again $c_0 = 1$, the tip's curvature reads

$$\kappa_0 = \partial_{xx} h(0, t) = \frac{c^3}{M^2} \sim \frac{t^3}{M^2}. \tag{4.4}$$

The vertical acceleration of the tip in R' is given by

$$a_t \Big|_{R'} = g(t) - \frac{2M^2}{t^5} + o(t^{-5}). \tag{4.5}$$

The asymptotic self-similar equation of the surface reads

$$-\frac{X^2}{M^2} + Z^2 = 1, \tag{4.6}$$

where $X = xt$ and $Z = z/t$, and we have shown that in two dimensions, the curvature scales like t^3 and the overshoot in acceleration undergone by the spike scales like t^{-5}.

5. Numerics

The theory is tested using a simplified two-dimensional version of the RM instability. This instability occurs when a shock wave impulsively accelerates an interface between two fluids with different densities. In the case we consider, the fluid is incompressible and an initial velocity profile is imposed at the free surface. In reality, this velocity profile would result from the impact of the shock wave on the free surface, a situation that corresponds to the experiment described by Antkowiak *et al.* (2007), for which the evolution time scale of the free surface is much larger than the acoustic time scale.

Initially, the fluid is at rest and the free surface is flat. An initial velocity potential $\varphi(x, y) = -5\cos(x)$ is imposed on the free surface and the subsequent evolution of the periodic surface (in the absence of a body force) is computed using a boundary integral method. Successive profiles of the free surface are shown in figure 2. We expect the theory to be valid once the jet is very elongated, i.e. when its vertical extension is much larger than the horizontal length scale. We can approximate the critical time t_0 after which the self-similar behaviour should be observed by $t_0 \approx L/2v_0$, where L is the wavelength and v_0 is the initial velocity. In our numerical simulation, $L = 2\pi$ and $v_0 = 5$, then $t_0 \approx \pi/5$.

The boundary integral method used to test the theory is a two-dimensional periodic method described in detail by Duchemin *et al.* (2005). Knowing φ on the free surface at a given time step, we solve a Fredholm equation of the second type

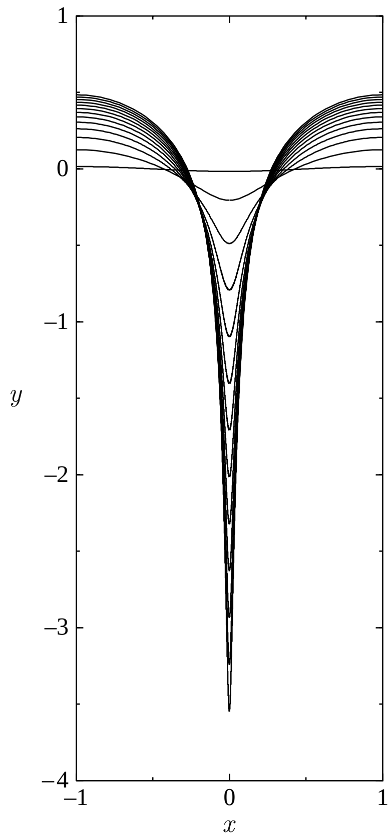

Figure 2. Computed successive profiles during RM instability, for different times t ranging from 0 to 1.2 and separated by 0.1. Horizontal and vertical coordinates have been rescaled according to π. As in the RT instability, a high-curvature spike develops at large time.

(the integral representation of equation (2.1)), which allows us to find the stream function on the surface and therefore the velocity of fluid particles. Then, the velocity potential is updated in time using Bernoulli equation, and the points are moved according to the kinematic condition on the surface. We emphasis the fact that a non-uniform mesh is used. Indeed, more collocation points are distributed at each time step around the spike than on the rest of the surface, in order to obtain an accurate representation of this region, in terms of geometry and dynamics.

Using this method, we were able to test the two-dimensional theory. Figure 3 shows the curvature of the tip as a function of time in a log–log plot. The long-dashed line corresponds to a fit of the curvature in t^3/M^2. A value of $1/M^2=1000$ is a good approximation for the prefactor, but it is worth mentioning that an error of $\pm 5\%$ on this prefactor can exist. Knowing the prefactor $1/M^2$ from the curvature, there remains no adjustable parameter in the self-similar shape of the surface and the acceleration of the spike. The dashed curve in figure 3 is the acceleration of the tip as a function of time in a log–log plot, and the long-dashed line corresponds to the scaling law $2M^2/t^5$, using again $1/M^2=1000$. Figure 4 shows the expected self-similar evolution of the spike. The collapsed profiles on the

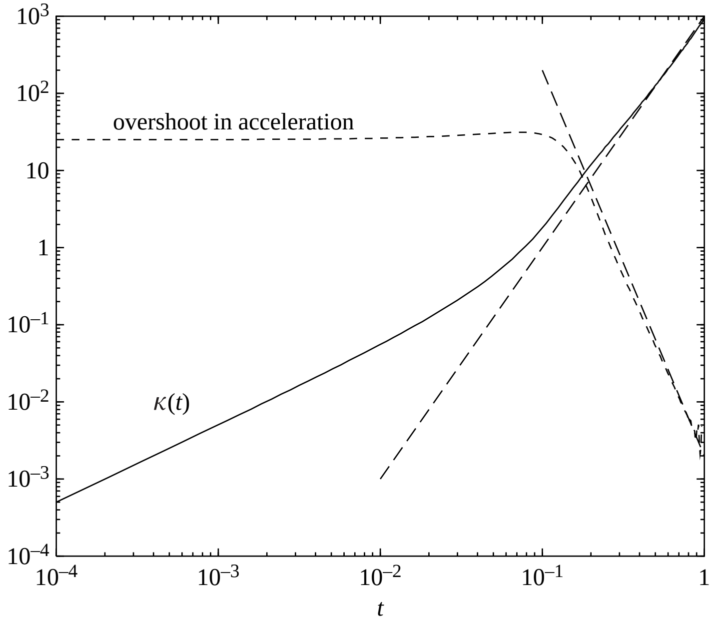

Figure 3. Curvature of the spike (solid line) and overshoot in acceleration (dashed line) as a function of time in a log–log plot. The long-dashed lines correspond to the predicted scaling laws t^3/M^2 and $2M^2/t^5$ for the curvature and the overshoot, respectively. The same value $M^2=1/1000$ has been used to draw these two curves.

r.h.s. of figure 4 confirm not only that the shape is asymptotically self-similar, but also that the self-similar shape is indeed the one predicted by the theory (4.6), and that the coefficient $1/M^2$ found in figure 3 is correct. Thanks to these numerical computations, we have been able to check the linearity in z of φ_z on a large part of the jet. This numerical check together with the self-similar behaviour are in favour of the use of equation (2.3) as an ansatz to describe the dynamics of the jet. Using a pure straining flow potential, we have shown that an inviscid thin jet undergoes an overshoot in acceleration scaling like t^{-4} in three dimensions and t^{-5} in two dimensions, that its curvature scales like t^2 in three dimensions and t^3 in two dimensions and finally that the shape of the jet is a hyperboloid in three dimensions, a hyperbola in two dimensions. The ansatz (2.3) remains to be validated in three dimensions and the self-similar behaviour and scaling laws to be confronted with accurate axisymmetric computations.

6. Conclusions

The constant M can be related to the physics of these instabilities. Indeed, in two dimensions, the asymptotic dimensional curvature reads $\kappa' \sim t'^3/\beta^2$, with $\beta^2 = M^2 L T^3$, where L and T are, respectively, the length and time scales used to non-dimensionalize the equations. On the other hand, the dimensional amplitude of the initial perturbation reads $\alpha = AL^2/T$, where A is the dimensionless amplitude. Eliminating T between β^2 and α, on which we have no control, we find $\beta^2 = (A^3M^2)L^7/\alpha^3$. Since A and M do not depend on the

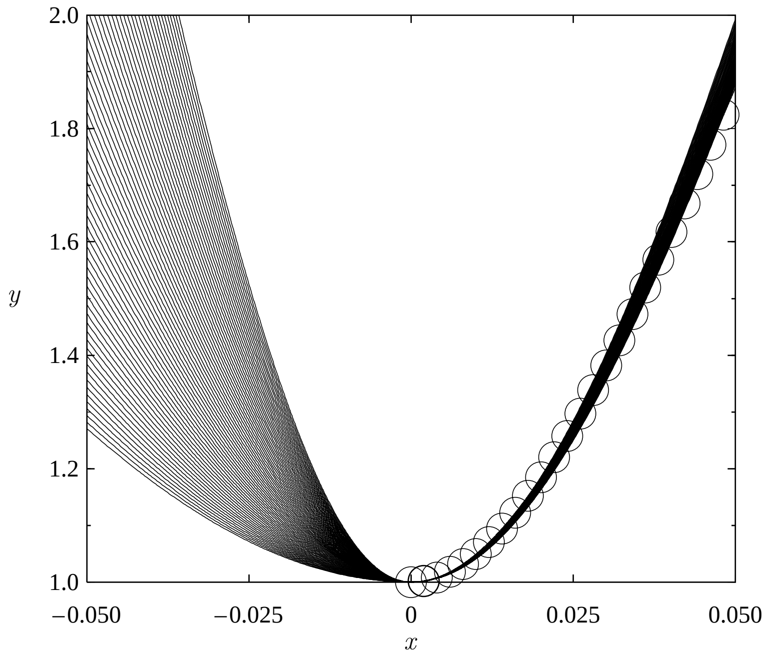

Figure 4. Self-similar structure of the tip: the interface profiles around the spike have been superimposed (and shifted vertically such that $y(0)=1$) on the l.h.s. for different times t ranging from 0.62 to 1.27. The r.h.s. shows the same curves rescaled by factors $1/t$ and t for the x and y coordinates, respectively, following the scaling behaviour predicted by the theory. The circles correspond to the self-similar equation of the surface (4.6), with $1/M^2=1000$ found from the curvature in figure 3.

physical quantities β^2, L and α, then A^3M^2 has to be a constant (numerically, by varying A, we find $A^3M^2 \simeq 8$) and the physical parameter β^2 is set by the choice of L and α. Equivalently for the RT instability in two dimensions, we find $\beta^2 = M^2L^{5/2}/g^{3/2}$. One of the most challenging studies remaining is the matching of the spike geometry and dynamics with the asymptotic bubble going up (cf. figure 2), which would give the value of M^2A^3 for any initial condition and would allow a global description of the nonlinear evolution of these instabilities. Moreover, an interesting future would be to include weak effects of surface tension in this theory.

From an experimental point of view, although the overshoot in acceleration is a very small nonlinear effect, experiments showing the self-similar structure of the tip could be designed. Indeed, as long as the initial acceleration is large enough, viscosity and surface tension effects can be neglected on a wide range of time. For instance, for the experiment described by Antkowiak *et al.* (2007), we can define a Bond number $Bo = \rho g R^2/\sigma$, where ρ is the liquid density; g is the impulsive acceleration; R is a relevant length scale (the diameter of the tube or the diameter of the emerging jet); and σ is the surface tension coefficient. On the surface of the jet, a viscous boundary layer will grow in time with a thickness equal to $\sqrt{\nu t}$, where ν is the dynamic viscosity and t is time. In the experimental set-up used by Antkowiak *et al.* (2007), the Bond number is not large enough to discard the effect of surface tension. It would be very interesting to design an experiment with a much larger impulsive acceleration, such that capillary and viscous effects could be neglected, in order to observe the self-similar behaviour of the tip and the overshoot in acceleration.

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