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SLOW DENSE GRANULAR FLOWS AS A SELF INDUCED PROCESS.

OLIVIER POULIQUEN*AND YOËL FORTERRE
IUSTI, Université de Provence, 5 rue Enrico Fermi,
13453 Marseille cedex 13, France

STÉPHANE LE DIZES
IRPHE, Université de Provence, 49 rue Joliot Curie
B.P. 146 13384 Marseille Cedex 13, France

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A simple model is presented for the description of steady uniform shear flow of granular material. The model is based on a stress fluctuation activated process. The basic idea is that shear between two particles layers induces fluctuations in the media that may trigger a shear at some other position. Based on this idea a minimum model is derived and applied to different configurations of granular shear flow.

1. Introduction

The description of flow of cohesionless granular material still represent a challenge [1]. In a collisional regime, when the medium is dilute and strongly agitated, hydrodynamic equations have been proposed by analogy with a molecular gas [2,3]. Assuming the collisions between particles to be instantaneous and inelastic, one can derive constitutive equations for the density, velocity and granular temperature (a measure of the velocity fluctuations). However, in many cases energy injected is not sufficient and the dissipation due to the inelastic collisions is so efficient that the medium does not stay in a collisional regime and flows in a so called dense regime. The particles experience multibody interactions and long lived contacts. The material can no longer be seen as a granular gas and there is a need for another description. Experiments on dense granular flows have been carried out in different configurations including shear cell [4,5,6], silo [8,9,10,11], flow down inclined planes [12,13,14], flow at the surface of a heap [15,16]. For these configurations precise information is now available about the velocity profiles, the fluctuations, and stresses that developed during the flows. However there is a lack of unified theoretical description.

*email: olivier@iusti.univ-mrs.fr

From the theoretical point of view, several approaches have been proposed to describe dense granular flows. A first attempt has been made to extend the kinetic theory of granular matter by introducing a shear rate independent term in order to incorporate friction ([17,18]). Another approach has been proposed recently by Bocquet et al [19]. They suggest to change the density dependence of the viscosity in order to take into account the fact that particles at high volume fraction are trapped in cages. However it is not clear that such approaches are still valid when interactions are not collisional. Savage [20] proposed a hydrodynamic model based on a fluctuating plasticity model. He obtained a hydrodynamic description close to the kinetic theory where the viscosity decreases with the temperature. Aranson et al [21] developed an empirical theory based on the idea of a mixture of fluid and a solid, the relative concentration of both constituents being driven by a Landau equation. This model seems to be efficient in describing non stationary effects like avalanche triggering. Mills et al [22] and Jenkins and Chevoir [23] described the granular fluid as a classical viscous fluid in which collective objects such as arches or columns are present, propagating the stresses in a non local way. They provide a non local description of the flow down inclined planes.

In this paper we present an alternative approach for describing slow dense granular flows. The model is based on a stress fluctuation activated process. The idea is that a shear somewhere in the material induces stress fluctuations in the medium which can in return help shearing somewhere else. A crude model of fluctuation activated process was previously proposed in the context of flow in a silo [10]. More recently, Debregeas et al [7] developed a fluctuation model to describe velocity profiles and correlations observed in a simple shear flow experiments on foams. In this paper we will present a simple model where the stress fluctuations are induced by the shear itself. We apply the model to different configurations.

2. Description of the model

The main idea of the model is sketched in Fig.1. If a shear motion occurs at a level z' between two layers, local rearrangements will occur and induce stress fluctuations in the assembly of particles. These stress fluctuations can help the material to yield somewhere else in z. The motion in z is then related to the fluctuations created by the level z'.

In order to apply this idea, we consider a stationary parallel shear flow of particles with a mean velocity $u(z)$ along x which depends only on the vertical coordinate z (Fig. 1). We call $P(z)$ and $\tau(z)$ respectively the pressure and the shear stress at the level z. The particles diameter is d. For simplicity, the model is written in terms of a stack of layers and only the direction z is considered.

The yield criterion in z can be expressed in term of a Coulomb criterion by introducing a friction coefficient μ : the material will spontaneously shear in z if the local shear stress $|\tau(z)|$ reaches the value $\mu P(z)$ otherwise nothing happens. The friction coefficient μ take into account both the friction between grains and

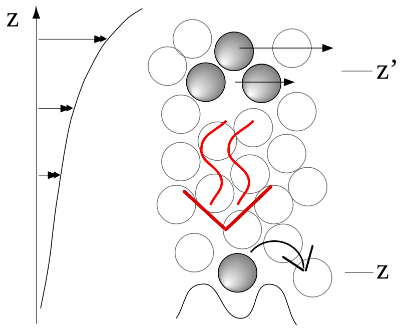

Fig. 1. Sketch of the self activated process: shear at a level z' induces fluctuations which can induce yielding in z.

the geometric entanglement. However, if stress fluctuations exist in addition to the mean stresses, the yield criterion can be locally reached even if the mean values of the shear and normal stresses do not verify the yield criterion.

This is for example the case when shear occurs at the level z' in the medium. If $\delta\sigma_{z'\to z}$ is the amplitude of the stress fluctuation induced by a shear in z' and measured at the level z, then this fluctuation will be sufficient to induce yielding in z if $\delta\sigma_{z'\to z} > (\mu P - |\tau|)$.

We then write that each time z' send a fluctuation, the particle in z will jump to the next hole on the right with a probability equal to the probability that the stress fluctuation is higher than the threshold: $\mathcal{P}\left[\delta\sigma_{z'\to z} > (\mu P - \tau)\right]$. By symmetry we can write that the probability to jump to the next hole on the left is $\mathcal{P}\left[\delta\sigma_{z'\to z} > (\mu P + \tau)\right]$.

We then write that the frequency at which z' send fluctuations is simply the shear rate amplitude at z': $\left|\frac{du}{dz}(z')\right|$. Stipulating that the event coming from different levels z' are uncorrelated, we can write the shear rate in z as a sum over z':

$$\frac{du}{dz}(z) = \sum_{z'} \left|\frac{du}{dz}(z')\right| \left(\mathcal{P}\left[\delta\sigma_{z'\to z} > (\mu P - \tau)\right] - \mathcal{P}\left[\delta\sigma_{z'\to z} > (\mu P + \tau)\right]\right) \qquad (2.1)$$

It is important to keep in mind that in expression (2.1) several assumptions have been made. First, it is assumed that there is no memory in the process: if the stress fluctuation received at some point does not reach the threshold, the system come back to its initial state and the next fluctuation will have to reach the same threshold to induce yielding. There is no stress accumulation.

A second assumption which is made is that the jumps are instantaneous. The time necessary for the particle to jump to the next hole (of the order of $d\sqrt{\rho/P}$) is negligible compare to the elapsed time between two successive fluctuations (du/dz^{-1}). This means that the model developed here will only apply for slow shear rates and quasi static deformations for which $du/dz\, d\sqrt{\rho/P} << 1$.

In order to express the probabilities in (2.1) we have to choose a probability distribution for the stress fluctuation $\delta\sigma_{z'\to z}$ induced by z' on z. A plausible assumption is that the amplitude of fluctuation is a decreasing function of the distance $(z'-z)$ and that it is maximum for $z=z'$ equal to some value $\delta\sigma_0(z')$. We choose in the following :

$$\delta\sigma_{z'\to z} = \frac{\delta\sigma_0(z')}{1+\beta(z-z')^2/d^2}$$

where β is a dimensionless parameter of the order of unity measuring the characteristic length in particle diameters over which the fluctuation decays. We tried other decreasing functions and the results described in this paper were qualitatively unchanged.

The amplitude $\delta\sigma_0(z')$ is a random variable with a mean which has to be of the order of the local pressure $P(z')$. Recent studies of the distribution of forces in granular piles suggest that the distribution is exponential over a large range of forces [24,25,26]. We then assume in the following that $\delta\sigma_0(z')$ follows an exponential distribution i.e. the probability $p\,d(\delta\sigma_0)$ to have the value $\delta\sigma_0$ within a range $d(\delta\sigma_0)$ is equal to:

$$p\,d(\delta\sigma_0) = \frac{1}{P(z')}\exp\left(-\frac{\delta\sigma_0}{P(z')}\right) d(\delta\sigma_0).$$

We can then easily show that the probability of jumps in expression (2.1) can also be expressed in term of an exponential:

$$\begin{aligned}\mathcal{P}\left[\delta\sigma_{z'\to z} > (\mu P-\tau)\right] &= \int_{(\mu P-\tau)}^{\infty} \frac{p}{1+\beta(z-z')^2/d^2} d(\delta\sigma_0) \\ &= \exp\left(\tfrac{(\mu P(z)-\tau(z))\left(1+\beta(z-z')^2/d^2\right)}{P(z')}\right) \qquad (2.2)\end{aligned}$$

It is interesting to note that the above expression is similar to an activated process where the role of the energy barrier would be played by $\mu P(z)-\tau(z)$ and the role the temperature by the mean stress fluctuation.

Finally, transforming the discrete sum in (2.1) as an integral, we can write the final expression for the shear rate as a function of the shear stress:

$$\begin{aligned}\frac{du}{dz}(z) = \frac{1}{d}\int \left|\frac{du}{dz}(z')\right| \Bigg(\exp\left(\frac{(\mu P(z)-\tau(z))\left(1+\beta(z-z')^2/d^2\right)}{P(z')}\right) - \\ \exp\left(\frac{(\mu P(z)+\tau(z))\left(1+\beta(z-z')^2/d^2\right)}{P(z')}\right)\Bigg)\, dz' \qquad (2.3)\end{aligned}$$

This integral equation gives a rheological law relating the stresses and shear rate. There are only two parameters in the model, μ the friction coefficient at the grain level and β the dimensionless extension of the fluctuations. In the following we solve this equation in different configurations. Note that if the shear keeps the same sign across the layer, eq. (2.3) is linear with respect to the shear rate. This means that a non zero shear rate profile will correspond to specific values of the stress distribution. This eigenvalue problem is solved numerically by discretizing the integral in eq.(2.3). Once the shear rate profile is known, the velocity profile is obtain by integration with the condition of zero velocity on the walls.

3. Uniform shear

Let first consider the simple rheological test of an infinite medium without gravity with an imposed constant shear : $\frac{du}{dz} = \Gamma = cte$. The pressure P is fixed to a constant. The equation 2.3 then indicates that non zero shear rate exists only if the shear stress τ satisfies:

$$\frac{e^{-(\mu-\tau/P)}}{\sqrt{\beta(\mu-\tau/P)}} - \frac{e^{-(\mu+\tau/P)}}{\sqrt{\beta(\mu+\tau/P)}} = \sqrt{\frac{\beta}{\pi}} \tag{3.1}$$

According to the model, a simple shear is then obtained for a specific value of the ratio τ/P. The material behaves like a frictional material with a coefficient of friction depending on the two parameters μ and β. For example, for the typical values $\mu = 0.7$ and $\beta = 2$, one find that $\tau/P = 0.42$. The macroscopic friction coefficient τ/P is then less than the microscopic one μ.

4. Silo

In this section we address the problem of a granular material flowing in a silo consisting of two parallel rough walls. Experiments have been carried out to measure the velocity profile in two dimensional and three dimensional configurations. The main result is that the shear is localized close to the rough walls in two shear bands whose thickness is between 5 and 10 particles diameters, not very sensitive to the distance between the walls [8,10,11]. However it has been shown that the thickness of the shear zones is affected when the silo is inclined from the vertical. The shear zone close to the bottom wall is larger and the other one thinner [10].

With the model presented in this paper it is possible to solve the problem for the flow between two rough planes. In this configuration the material is bounded by two walls at a distance L. In the model the wall are simply the first and last row of particles at $-L/2$ and $L/2$ which becomes the limits of the integral in 2.3. According to the equilibrium relations the stresses satisfy :

$$\frac{dP}{dz} = -\rho g \cos(\theta), \frac{d\tau}{dz} = -\rho g \sin(\theta) \tag{4.1}$$

where θ is the angle between the walls and horizontal, ρ is the density of the grain packing and g the gravity. Hence, $P(z) = -\rho g \cos(\theta) z + P_0$ and $\tau(z) = -\rho g \sin(\theta) + \tau_0$ where P_0 and τ_0 are constant which are a priori unknown. The constants are determined by the eigenvalue problem of eq. (2.3). As we expect the shear rate to change sign in the silo, we have to solve eq. (2.3) for both signs of the absolute value $\left|\frac{du}{dz}(z')\right|$ and then match the two solutions to get the correct solution

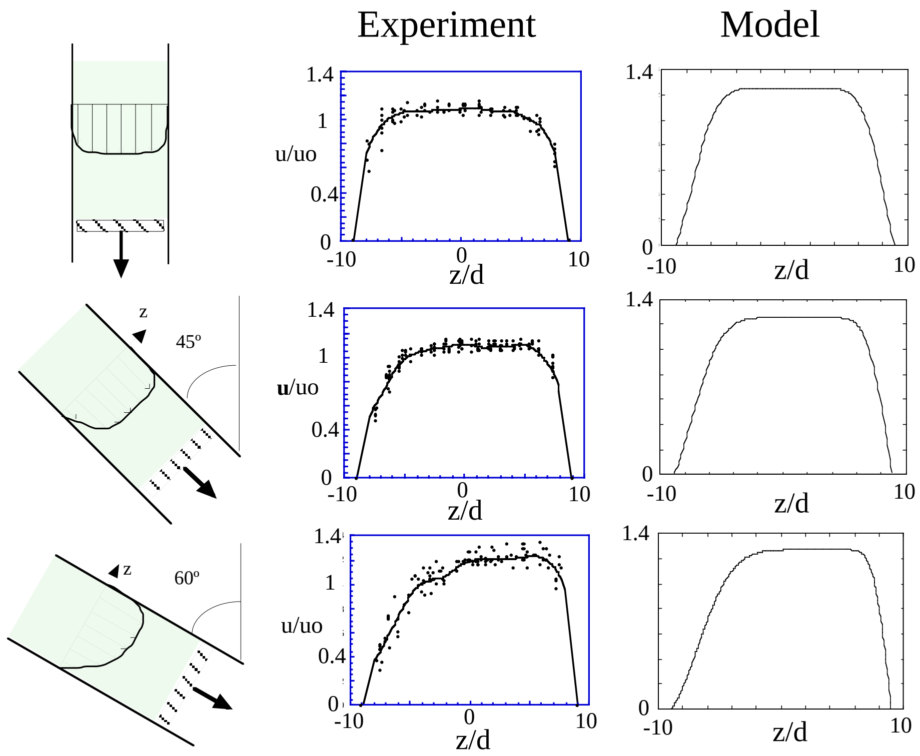

Fig. 2. Velocity profiles observed in inclined silo in 2d experiments (Data from [10]) and predicted by the model ($\mu = 0.7$ $\beta = 2$)

of eq. (2.3). This matching condition allows to find the two constant P_0 and τ_0 and the associated shear rate profile.

The predicted velocity profiles are presented in Fig. 2 for three different inclinations of the silo. We have plotted both the velocity profile observed in the experiments and predicted by the model. For the vertical case, the model predicts the existence of shear zones localized close to the walls as observed in the experiments. The width of the shear zone varies slightly with the distance between the rough walls but is of the order of five particle diameters as shown in Fig. 3. When the silo is inclined, the velocity profile is asymmetric and the shear zone is larger on the bottom side than on the upper side as observed in the experiments.

5. Flow down inclined planes

We then address the problem of the flow down a rough plane. The two control parameters are in this case the inclination θ of the plane and the thickness of the flow h (Fig. 4). Numerical and experimental studies have been carried out giving

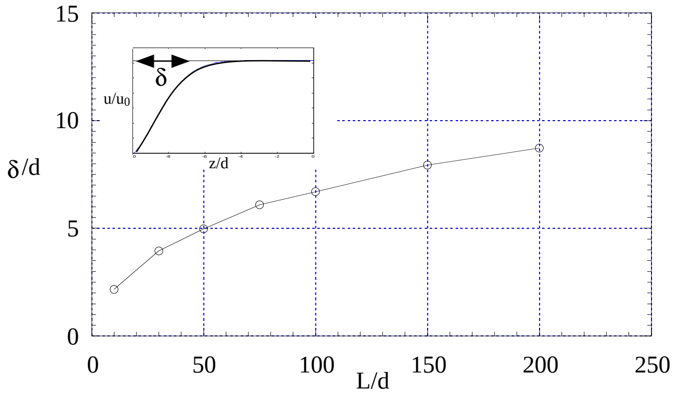

Fig. 3. Thickness of the shear zone as a function of the distance between the walls ($\mu = 0.7$ $\beta = 2$)

information about the properties of steady uniform flows [13,12,27]. The linear model developed in this paper is only valid for quasi-static deformations, which means that we will not be able to describe the fully developed flow down inclines. A non linear extension of the model is under development and will be the subject of another study. However, from the present model we can get information about the onset of flow. It has been shown experimentally and numerically that for a given inclination θ, there exist a minimum thickness $h_{stop}(\theta)$ below which no steady uniform flow is observed. The present model is able to predict such a behavior.

In this configuration the material is bounded by a rough bottom which in the model is simply the first row of particles, and a free surface at $z = h$. In order to use the same definition for the layer thickness as in the experimental work i.e. the fixed layer is not counted, the integration domain in eq. 2.3 is from $-d$ up to h. The stress distribution is derived from the equilibrium and from no stress boundary condition at the free surface: $P = \rho g cos(\theta)(h - z)$, and $\tau = \rho g sin(\theta)(h - z)$.

From eq.2.3 it comes out that for a given thickness h, a non zero shear rate profile exists only for a given value of the inclination. This critical inclination is a function of the thickness of the layer h as shown in Fig. 4. The model predicts that in order to flow, a thin layer has to be more inclined than a thick layer. This result is reminiscent of the onset of flow observed in the experiments. This behavior can be easily understood in the present model: for thin layer, the integral in eq.2.3 extends over a narrower region which means that there are less sources of fluctuation to help yielding. The inclination has then to be higher in order to be closer to the yielding threshold.

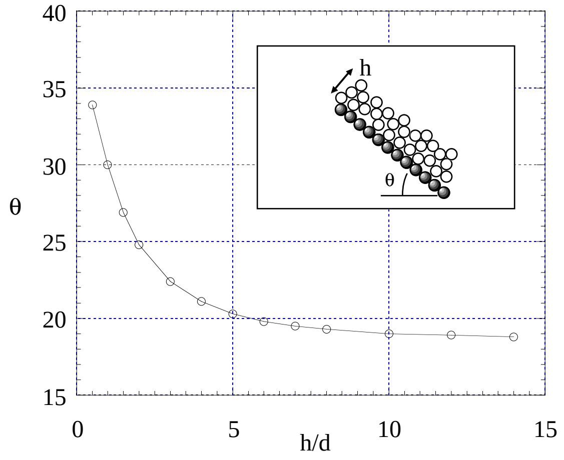

Fig. 4. Critical angle at which a flow down an inclined plane is possible as a function of the thickness of the layer h ($\mu = 0.7$ $\beta = 2$)

6. Shear between plates

The last example we want to address is the simple shear experiment. Recently careful measurements of the velocity profile have been performed in Taylor Couette cell where the granular material is sheared between two concentric cylinders in 2D or 3D geometry [4,5,6]. The shear is localized close to the moving walls and extends up to 5 or 10 particles diameter in the bulk. The shear zone is associated with a slightly lower solid fraction.

We have solved our model for the shear flow between two plates, one being fixed, the other one moving at a velocity 1. The pressure P and the shear stress τ are constant across the layer. Again, the linearity of the problem imposes that a solution is found only for a specific value of the ratio τ/P. The corresponding velocity profile is plotted in Fig. 5a for a gap between the plate equal to $40d$. The velocity profile we obtained is not localized close to the walls : the shear is distributed over the whole material. The phenomena of localization in simple shear flow is then not predicted with the simple model presented here.

However, several modifications of the model give rise to a localization. First our configuration is plane whereas most of the experiments are carried out in cylindric geometry. A small asymmetry introduced for example in the pressure distribution to take into account the cylindric geometry, gives rise to a localized shear band. However, in this case the thickness of the shear zone depends on the asymmetry which does not seem to be observed in experiments.

A more convincing improvement of the model consists in taking into account boundary effects in the perturbation function $\delta\sigma_{z' \to z}$. The stress fluctuation induced by a shear close to the wall is not the same as the fluctuation induced when the shear occurs far from the wall in the bulk. The wall being rigid, we have a zero displacement condition. Qualitatively we then expect that a shear close to a wall induces higher fluctuation than a shear in the bulk, as part of the fluctuation is reflected by the wall. A simple way to take this wall effect into account is to add an additional source of fluctuation on the other side of the wall, image from the real one. In the case of the shear cell where we have two rigid walls at position 0 and H, we then write the perturbation $\delta\sigma_{z'\to z}$ as the sum of three terms corresponding to a source in z', a source in $z'' = -z'$ and one in $z''' = 2H - z'$.

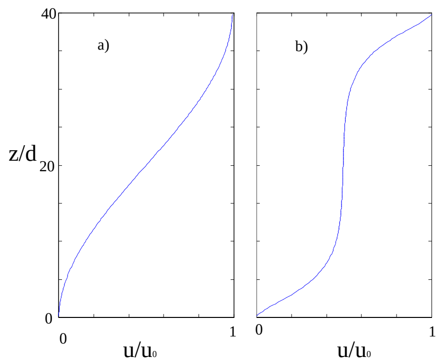

Fig. 5. Velocity profile predicted for the shear between two plates. a) simple model b) model with image sources of fluctuation.

Using the image sources do not qualitatively change any of the results presented above for the silo or the inclined plane. However, for the shear between two walls, the solution shows two shear zones localized close to the walls as shown in Fig. 5b. Therefore, more dangerous fluctuations close to the walls induce a localization of the shear bands.

7. Discussion and conclusions

In this paper we have presented a simple model for slow sheared granular flows which is based on stress fluctuations. The process we describe is self induced as the shear induces fluctuations which in return induce shear. As a result we obtain an integral rheological law, where the shear rate at a position is related to the shear rate in the vicinity. Quasi static flow like flows between vertical plates or in shear cells are well described by the model. It is also able to predict the onset of flow for the inclined plane configuration.

However, the model is still very crude. First it is one dimensional: the stress fluctuations are described by the z coordinate. A careful 3d analyses is certainly necessary to better describe the fluctuations and shear induced motions. A second improvement we are working on is the finite duration of the jump. When yielding is activated at some place by a fluctuation, it takes a finite time for the particle to go to the next hole. Taking into account this time delay should allow to describe more rapid flows like inclined chute flows.

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References

[1] J. Rajchenbach, *Advances in Physics* **49**, 229 (2000).
[2] C. K. K. Lun *et al*, *J. Fluid Mech.* **140**, 223 (1984).
[3] I. Goldhirsch, *Chaos* **9**, 659 (1999).
[4] C.T. Veje , D.W. Howell and R.P. Behringer , *Phys. Rev. E* **59** 739 (1999).
[5] W. Losert,L. Bocquet ,T.C. Lubensky and J.P. Gollub ,*Phys. Rev. Lett.* **85** 1428 (2000).
[6] D.M. Mueth ,G.F. Debregeas ,G.S. Karczmar , P.J. Eng, S.R. Nagel , and H.M. Jaeger , *Nature*, **406** 385 (2000).
[7] G. Debregeas , H. Tabuteau and J. M. di Meglio , to be published in *Phys. Rev. Lett.* cond-mat/0103440.
[8] R.M. Nedderman and C. Laohakul , *Powder Technology* **25** 91 (1980).
[9] V.V.R. Natarajan , M.L. Hunt and E.D. Taylor ,*J. Fluid Mech.*, **304** 1 (1995).
[10] O. Pouliquen and R. Gutfraind ,*Phys. Rev.E*, **53** 552 (1996).
[11] F. Chevoir *et al* in *Powders and Grains*, ed. Y. Kishino, (Lisse, Swets and Zeitlinger (2001)) 399-402.
[12] F. Chevoir *et al*, in *Powder and Grains 2001*, ed. Y. Kishino, (Lisse, Swets and Zeitlinger (2001)) p.373-376.
[13] O. Pouliquen, Phys. Fluids **11**, 542 (1999).
[14] A. Daerr and S. Douady, *Nature* **399**, 241 (1999).
[15] P.A. Lemieux and D.J. Durian , *Phys. Rev. Lett.*, **85** 4273 (2000).
[16] D. V. Khakhar , A. V. Orpe , P. Andersen , J. M. Ottino , *J. Fluid Mech.*, **441** 255 (2001).
[17] S.B. Savage , In *Mechanics of granular Materials: new Models and constitutive relations*, eds. J. T. Jenkins and M. Satake (Elsevier, Amsterdam, 1983).
[18] P. C. Johnson, P. Nott and R. Jackson, *J. Fluid Mech.* **210**, 501 (1990).
[19] L. Bocquet *et al* To be published in *Phys. Rev. E*, cond-mat/0012356 (2001).
[20] S. B. Savage, *J. Fluid Mech.* **377**, 1 (1998)
[21] I. S. Aranson and L. S. Tsimring , *Phys. Rev. E* **64**, 020301 (2001).
[22] P. Mills, D. Loggia and M. Tixier , *Europhys. Lett.* **45**, 733 (1999).
[23] J.T. Jenkins , F. Chevoir,“ Dense plane flows of frictional spheres down a bumpy , frictional incline”, preprint (2001).
[24] C. H. Liu *et al*, *Science* **269** 513 (1995).
[25] F. Radjai, M. Jean, J.j. Moreau et S. Roux, *Phys. Rev. Lett.* **77** 274 (1996).
[26] D. M. Mueth, H. Jaeger et S. R. Nagel, *Phys. Rev. E* **57** 3164 (1998).
[27] L. E. Silbert *et al*, preprint cond-mat/0105071 (2001).