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Note on 'Multi-frequency Craik–Criminale solutions of the Navier–Stokes equations' by B. R. Fabijonas and D. D. Holm

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In a recent paper (*J. Fluid Mech.* vol. 506, 2004, p. 207), B. R. Fabijonas and D. D. Holm claim that they have found a general method to construct new solutions of the Navier–Stokes equations from any known base flow solution. In this note, we argue that Fabijonas & Holm's solutions are very special in character. Although they can be defined in all space, they satisfy the Navier–Stokes equations on a single fixed trajectory of the chosen base flow. We show that this limits the usefulness of the solution and the applicability of the method. In particular, it is demonstrated that, in general, the iterative 'WKB-bootstrapping' algorithm designed by the authors cannot be applied after the first iteration. We also show that a second iteration is possible only if the base flow satisfies strong constraints. The consequence of these constraints is that no extension of the Craik–Criminale solutions to multiple frequencies is found to be possible. By applying Fabijonas & Holm's construction to a simple model equation, we demonstrate that their solution can also predict (unphysical) behaviours which cannot be reproduced by any global solution.

1. Introduction

In 1986, Bayly and Craik & Criminale independently constructed new solutions of Euler and Navier–Stokes equations by considering the superimposition of plane wave disturbances on a base flow with spatially uniform velocity gradient. They demonstrated that if the phase of the disturbance is advected by the base flow, the nonlinear interactions vanish and an exact nonlinear solution is formed. In 1991, Friedlander & Vishik and Lifschitz & Hameiri independently developed a short-wavelength instability theory to study the evolution of localized inviscid disturbances to any base flow. They showed that if the disturbance is sufficiently localized, it is advected along the flow trajectory, and governed by a set of linearized equations formally similar to those obtained for flows with uniform velocity gradient. A review of both approaches may be found in Friedlander & Lipton-Lifschitz (2003).

In a recent paper, Fabijonas & Holm (2004, hereafter referred to as FH) attempted to construct new Navier–Stokes solutions from any exact solution by applying iteratively the procedure used in Craik & Criminale. In this note, the limitations associated with the method are discussed. We show that no new solutions can be obtained.

The framework of the analysis is the following. Consider a base flow, a solution of the Navier–Stokes equations, given by its velocity and pressure fields $\boldsymbol{U}(\boldsymbol{x}, t)$

and $P(\boldsymbol{x}, t)$. Finite-amplitude disturbances $\boldsymbol{U}'(\boldsymbol{x}, t)$ and $P'(\boldsymbol{x}, t)$ to the base flow are governed by

$$\mathrm{D}_t \boldsymbol{U}' + \mathscr{L}\,\boldsymbol{U}' + (\boldsymbol{U}' \cdot \nabla)\boldsymbol{U}' + \nabla P' = \nu \Delta \boldsymbol{U}', \quad \nabla \cdot \boldsymbol{U}' = 0, \tag{1.1}$$

where $\mathrm{D}_t = \partial_t + \boldsymbol{U} \cdot \nabla$ and $\mathscr{L}(\boldsymbol{x}, t) = \nabla \boldsymbol{U}$ is the base flow velocity gradient.

Consider now a given trajectory $\mathscr{Q}_{\boldsymbol{x}_0}$ of the base flow $\boldsymbol{U}$ defined by $\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{x}_0)$ where $\mathrm{d}\boldsymbol{\chi}/\mathrm{d}t = \boldsymbol{U}(\boldsymbol{\chi}(t, \boldsymbol{x}_0), t)$ with $\boldsymbol{\chi}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$. FH considered expressions for $\boldsymbol{U}'$ and P' of the form

$$\boldsymbol{U}'(\boldsymbol{x}, t) = {\sum_m}' \mu^{|m|} \boldsymbol{a}_m(t) \mathrm{e}^{\mathrm{i}m\beta\Phi(\boldsymbol{x},t)}, \quad P'(\boldsymbol{x}, t) = {\sum_m}' \mathrm{i}\mu^{|m|} p_m(t) \mathrm{e}^{\mathrm{i}m\beta\Phi(\boldsymbol{x},t)}, \tag{1.2}$$

where $\mu^{|m|}$ and β are real constants and the summation runs from $-\infty$ to $+\infty$ with $m \neq 0$. They furthermore assumed that the phase $\Phi(\boldsymbol{x}, t)$ is passively advected:

$$\mathrm{D}_t \Phi = 0 \tag{1.3}$$

and satisfies

$$\Phi(\boldsymbol{x}, t) = \boldsymbol{k}(t) \cdot \boldsymbol{x} + \delta(t). \tag{1.4}$$

FH showed that under these assumptions, expressions (1.2) with (1.4) satisfy equations (1.1) at $\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{x}_0)$ when

$$\dot{\boldsymbol{k}} = -\mathscr{L}^T(\boldsymbol{\chi}(t, \boldsymbol{x}_0), t)\boldsymbol{k}, \tag{1.5a}$$

$$\dot{\boldsymbol{a}}_m + \mathscr{L}(\boldsymbol{\chi}(t, \boldsymbol{x}_0), t)\boldsymbol{a}_m = m\beta p_m \boldsymbol{k} - \nu m^2 \beta^2 |\boldsymbol{k}|^2 \boldsymbol{a}_m, \tag{1.5b}$$

$$\boldsymbol{k} \cdot \boldsymbol{a}_m = 0, \tag{1.5c}$$

where $\dot{f}(t) = \mathrm{d}f/\mathrm{d}t$. In their analysis, each velocity amplitude $\boldsymbol{a}_m(t)$ evolves independently. Therefore in the rest of this note, we omit the subscripts m.

2. Main limitations of the construction

In this section, we shall show that FH's construction has two main limitations:

(*a*) the construction requires knowledge of an exact global solution $\boldsymbol{U}$;

(*b*) the new velocity field $\boldsymbol{U} + \boldsymbol{U}'$ satisfies the Navier–Stokes equations on a single fixed trajectory of $\boldsymbol{U}$, that is at each instant in a single point of space only. In particular, it is not a global solution.

Let us first consider (*b*). To prove this statement, it is useful to point out the main hypothesis which permits the reduction, in a general framework, of the perturbation equation (1.1) to (1.5): the functions $\boldsymbol{a}(t)$, $p(t)$ and $\boldsymbol{k}(t)$ appearing in expressions (1.2) and (1.4) must be independent of spatial variables. This hypothesis is clearly mentioned in FH, but the authors also remark that $\boldsymbol{a}(t)$, $p(t)$ and $\boldsymbol{k}(t)$ depend parametrically on the fixed streamline parameter $\boldsymbol{x}_0$, as it can be seen in (1.5). The authors argue that because this parameter can be arbitrarily chosen, expression (1.2) provides a solution in the entire Lagrangian space, that is, it provides a global solution. We claim that this is not correct because the parameter $\boldsymbol{x}_0$ appearing in (1.5) cannot be considered as a Lagrangian variable.

If $\boldsymbol{x}_0$ were a Lagrangian variable, the functions $\boldsymbol{a}$, p and $\boldsymbol{k}$ obtained from (1.5) would have to be considered as Lagrangian expressions. This would have important consequences. We know that, in an incompressible flow, the transformation $\boldsymbol{x}_0 \rightarrow \boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{x}_0)$ is a one-to-one mapping between the space of the initial configuration (the Lagrangian space) and the physical space at time t (the Eulerian space). Each field therefore possesses both an Eulerian representation $f^E(\boldsymbol{x},t)$ and a Lagrangian representation $f^L(t,\boldsymbol{x}_0)$ which are connected to each other by

$$f^L(t,\boldsymbol{x}_0) = f^E(\boldsymbol{\chi}(t,\boldsymbol{x}_0),t), \quad f^E(\boldsymbol{x},t) = f^L(t,\boldsymbol{\chi}^{-1}(\boldsymbol{x},t)). \tag{2.1}$$

Accordingly, Eulerian and Lagrangian gradients are also linked by

$$\nabla_{\boldsymbol{x}_0} = \mathscr{J}^T\nabla, \quad \nabla = (\mathscr{J}^{-1})^T\nabla_{\boldsymbol{x}_0}, \tag{2.2}$$

where $\mathscr{J}(t,\boldsymbol{x}_0)=\nabla_{\boldsymbol{x}_0}\boldsymbol{\chi}(t,\boldsymbol{x}_0)$ is the Jacobian matrix of the mapping $\boldsymbol{x}_0 \rightarrow \boldsymbol{\chi}(t,\boldsymbol{x}_0)$. These relations can be used to compute the action of an Eulerian gradient on a Lagrangian expression. In particular, (2.1) and (2.2) show that a Lagrangian expression is independent of spatial variables if and only if its Eulerian representation is also independent of spatial variables. In short, $f^L(t)\equiv f^E(t)\equiv f(t)$ if the field f is spatially uniform. This means that $\boldsymbol{a}$, p and $\boldsymbol{k}$ cannot depend on a Lagrangian variable if they are independent of spatial variables, as was assumed. Consequently, the streamline parameter $\boldsymbol{x}_0$ is not a Lagrangian variable.

To emphasize the difference between this parameter and the Lagrangian variable, that we continue to call $\boldsymbol{x}_0$, we assume that the streamwise parameter is fixed at a value $\boldsymbol{x}_0^{(0)}$. Expression (1.2) can therefore be considered as an Eulerian expression which depends on a parameter $\boldsymbol{x}_0^{(0)}$. But, by construction, this expression satisfies the perturbation equation (1.1), at each instant t, in a single point of space only: the moving point given by $\boldsymbol{x}=\boldsymbol{\chi}(t,\boldsymbol{x}_0^{(0)})$. By using (2.1), a Lagrangian representation of (1.2) can also be obtained by changing $\boldsymbol{x}$ to $\boldsymbol{\chi}(t,\boldsymbol{x}_0)$. In that case, the Lagrangian version of (1.1) obtained by using (2.2) is satisfied by this Lagrangian expression at a single Lagrangian point: the point which corresponds to the fixed streamline parameter $\boldsymbol{x}_0=\boldsymbol{x}_0^{(0)}$. Whatever the chosen representation, the solution obtained by FH's construction is therefore not a global solution in general, as it satisfies the perturbation equation along a single trajectory, that is at each instant in a single point of space only.

In § 6, we shall see for a simpler equation that this point-wise solution may not be related to any global solution. In particular, we shall demonstrate that it can predict unphysical behaviour of exponential growth whereas all the global solutions are damped. For the Navier–Stokes equations, other arguments are provided in the conclusion.

Let us now consider (*a*). This condition is explicitly mentioned in FH as $\boldsymbol{U}$ is assumed to be a base flow. This is necessary in order to reduce the Navier–Stokes equations to the system (1.1). It is also needed to be able to compute the flow trajectories and therefore to define the different expressions.

3. Consequences of the 'WKB-bootstrapping' algorithm

In this section, we argue that the two limitations described in the previous section make the iterative 'WKB-bootstrapping' inapplicable. The principle of the 'WKB-bootstrapping' algorithm is the following. We start from a solution $\boldsymbol{U}^{(0)}$, then apply the construction described above to obtain a new solution $\boldsymbol{U}^{(1)}=\boldsymbol{U}^{(0)}+\mu^{(1)}\boldsymbol{a}^{(1)}\sin(\beta^{(1)}\Phi^{(1)})$ on a trajectory of $\boldsymbol{U}^{(0)}$. Then, we apply the construction again to $\boldsymbol{U}^{(1)}$ to obtain a new solution $\boldsymbol{U}^{(2)}=\boldsymbol{U}^{(1)}+\mu^{(2)}\boldsymbol{a}^{(2)}\sin(\beta^{(2)}\Phi^{(2)})$ on a trajectory of $\boldsymbol{U}^{(1)}$ and so on. With this algorithm, FH argued that they could *a priori* form new solutions at each step provided that the phases are incommensurate. The problem is that after the first iteration, $\boldsymbol{U}^{(1)}$ is a solution on a single fixed trajectory of $\boldsymbol{U}^{(0)}$, that is at each instant in a single point of space only. In particular, this does not satisfy condition (a) which permits the reduction of the Navier–Stokes equations to equation (1.1) and the flow trajectory of $\boldsymbol{U}^{(1)}$ to be obtained. Therefore, we cannot perform the second iteration which gives $\boldsymbol{U}^{(2)}$ and the algorithm stops. To perform the second iteration, $\boldsymbol{U}^{(1)}$ must satisfy condition (a). We shall show in the next section that this considerably limits the possibilities for $\boldsymbol{U}^{(1)}$.

4. Necessary conditions to form a global solution

In this section, we provide a few necessary conditions for (1.2) to be a global solution. We have seen in the previous section that this is necessary to apply the bootstrapping algorithm. Thus, we assume that expression (1.2) with (1.3), (1.4) and (1.5) is not only a solution to (1.1) at the point $\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{x}_0^{(0)})$ but also in its neighbourhood. This implies that the spatially independent functions $\boldsymbol{k}$, $\boldsymbol{a}$ and p satisfy the system

$$\dot{\boldsymbol{k}} = -\mathscr{L}^T(\boldsymbol{x}, t)\boldsymbol{k}, \qquad (4.1a)$$

$$\dot{\boldsymbol{a}} + \mathscr{L}(\boldsymbol{x}, t)\boldsymbol{a} = \beta p \boldsymbol{k} - \nu\beta^2|\boldsymbol{k}|^2\boldsymbol{a}, \qquad (4.1b)$$

$$\boldsymbol{k} \cdot \boldsymbol{a} = 0, \qquad (4.1c)$$

not only at $\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{x}_0^{(0)})$ but in its neighbourhood. We now analyse the constraints that these hypotheses imply on the base flow.

Assume that the base flow has a non-uniform velocity gradient (otherwise we recover the Craik & Criminale analysis), then at least one of the matrices $\partial_x\mathscr{L}$, $\partial_y\mathscr{L}$ or $\partial_z\mathscr{L}$ is non-zero, say $\partial_x\mathscr{L}(\boldsymbol{x}, t) = \mathscr{L}_x(\boldsymbol{x}, t)$. Differentiating (4.1$a$) and (4.1$b$) with respect to x leads to

$$\mathscr{L}_x^T\boldsymbol{k} = 0, \quad \mathscr{L}_x\boldsymbol{a} = 0. \qquad (4.2)$$

Similar equations are obtained by differentiating with respect to any other spatial variable. The existence of non-zero solutions to these equations implies constraints on the base flow. Indeed, these equations mean that $\boldsymbol{k}$ and $\boldsymbol{a}$ are respectively the left and right eigenvectors associated with the null eigenvalue of $\mathscr{L}_x$. But the null eigenvalue must be at least of multiplicity two since $\boldsymbol{k}$ and $\boldsymbol{a}$ are orthogonal due to (4.1c). Moreover, since incompressibility implies $\mathrm{tr}\mathscr{L}_x = 0$, the matrix $\mathscr{L}_x(\boldsymbol{x}, t)$ can only have null eigenvalues. The same conclusion is also obtained for any other spatial derivative of $\mathscr{L}$. We suggest that this provides an important constraint which strongly limits the applicability of the method.

An illustration is easily provided using two-dimensional axisymmetrical flows. Consider the base flow of velocity field $\boldsymbol{U}(x, y) = (-y\Omega(r), x\Omega(r), 0)^T$, where $r = (x^2 + y^2)^{1/2}$ is the radial coordinate and $\Omega(r)$ the angular velocity. The base flow velocity gradient and its partial derivatives $\mathscr{L}_x$ and $\mathscr{L}_y$ may be easily computed, and it may be shown that their eigenvalues vanish identically around the trajectory $r = r_0$, if and only if $\Omega'(r_0) = 0$. It follows that the only two-dimensional axisymmetrical flow from which we can construct a global solution by FH's construction is solid body rotation, that is the flow which was initially considered by Kelvin (1880).

In § 6, we prove for a simplified model equation that the only 'base flow' from which we can construct non-trivial global solutions correspond to the case studied by Craik & Criminale. We suspect this conclusion to be also true for the Navier–Stokes equations but we have not been able to prove it.

5. Craik–Criminale solution

In their paper, FH also attempted to extend Craik–Criminale solution by applying the WKB-bootstrapping algorithm. We now show that no extension has here been made. The Craik–Criminale solution is the superimposition of a uniform flow $\boldsymbol{u}^{(0)}(\boldsymbol{x},t)=\mathscr{L}(t)\boldsymbol{x}$ and a Kelvin mode of the form

$$\boldsymbol{u}^{(1)}(\boldsymbol{x},t)=\mu^{(1)}\boldsymbol{a}^{(1)}(t)\sin\left(\beta^{(1)}\boldsymbol{k}^{(1)}(t)\cdot\boldsymbol{x}\right). \tag{5.1}$$

As demonstrated by Craik & Criminale, $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}$ is indeed an exact global solution of the Navier–Stokes equations provided that $\dot{\mathscr{L}}+\mathscr{L}^2$ is symmetric (with $\mathrm{tr}\mathscr{L}=0$), and that $(\boldsymbol{k}^{(1)},\ \boldsymbol{a}^{(1)})$ solve (4.1*a–c*). The first iteration of the algorithm provides a perturbation $\boldsymbol{U}'=\boldsymbol{u}^{(2)}$ to this base flow of the form

$$\boldsymbol{u}^{(2)}(\boldsymbol{x},t)=\mu^{(2)}\boldsymbol{a}^{(2)}(t)\sin\left(\beta^{(2)}\boldsymbol{k}^{(2)}(t)\cdot\boldsymbol{x}\right), \tag{5.2}$$

which satisfies equation (1.1) on a single trajectory of $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}$. This expression has already been obtained by Lifschitz & Fabijonas (1996) who also demonstrated that it can grow exponentially. Note, however, that Lifschitz & Fabijonas were able to demonstrate that (5.2) is the leading-order expression, in the limit of large $\boldsymbol{k}^{(2)}$, of an exact linear perturbation to $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}$.

FH went one step further by applying the algorithm to the new solution $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}+\boldsymbol{u}^{(2)}$ for a case where $\boldsymbol{u}^{(2)}$ is bounded. However, we argue here that this is not allowable; the new solution $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}+\boldsymbol{u}^{(2)}$ does not satisfy the Navier–Stokes outside the trajectory fixed at the previous step. As mentioned above, in order to apply the algorithm, $\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}+\boldsymbol{u}^{(2)}$ must be a base flow, that is an exact global solution. We shall now show that this implies that the constraint that $\boldsymbol{u}^{(2)}$ must be an harmonic of $\boldsymbol{u}^{(1)}$. The necessary conditions (4.2), obtained in the previous section, can be applied to the present case by taking $\boldsymbol{U}=\boldsymbol{u}^{(0)}+\boldsymbol{u}^{(1)}$. As

$$\mathscr{L}_x=\mu^{(1)}\left(\beta^{(1)}\right)^2k_x^{(1)}\boldsymbol{a}^{(1)}\otimes\boldsymbol{k}^{(1)}\sin\left(\beta^{(1)}\boldsymbol{k}^{(1)}\cdot\boldsymbol{x}\right), \tag{5.3}$$

equations (4.2) lead to the conditions

$$\boldsymbol{k}^{(2)}\cdot\boldsymbol{a}^{(1)}=\boldsymbol{k}^{(1)}\cdot\boldsymbol{a}^{(2)}=0, \tag{5.4}$$

which are in addition to those given by (4.1*c*):

$$\boldsymbol{k}^{(1)}\cdot\boldsymbol{a}^{(1)}=\boldsymbol{k}^{(2)}\cdot\boldsymbol{a}^{(2)}=0. \tag{5.5}$$

It is now easy to prove that $\boldsymbol{k}^{(2)}=C\boldsymbol{k}^{(1)}$ where C is a constant independent of time. For this purpose, assume that there is a time interval in which $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ are not colinear. Then, since both $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ are perpendicular to both vectors, they must be colinear, that is $\boldsymbol{a}^{(2)}=D(t)\boldsymbol{a}^{(1)}$. By manipulating equation (4.1*b*) written for $\boldsymbol{a}^{(1)}$ and $\boldsymbol{a}^{(2)}$ respectively, we obtain

$$\left(\dot{D}+\nu D\left(\left(\beta^{(2)}\right)^2\left|\boldsymbol{k}^{(2)}\right|^2-\left(\beta^{(1)}\right)^2\left|\boldsymbol{k}^{(1)}\right|^2\right)\right)\boldsymbol{a}^{(1)}=\beta^{(2)}p^{(2)}\boldsymbol{k}^{(2)}-D\beta^{(1)}p^{(1)}\boldsymbol{k}^{(1)}. \tag{5.6}$$

Owing to conditions (5.4) and (5.5), both sides must cancel separately, which in particular implies that $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ are colinear, in contradiction with our initial assumption. Thus, we always have $\boldsymbol{k}^{(2)}(t)=C(t)\boldsymbol{k}^{(1)}(t)$. Finally, from (4.1*a*) written for $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ we obtain $\dot{C}=0$, and therefore the results for the wavevectors we were looking for. In terms of solutions, this means that the secondary disturbance $\boldsymbol{u}^{(2)}$ is just an harmonic of $\boldsymbol{u}^{(1)}$.

The consequence of this result is that no tertiary instability is obtained by this method. More generally, we have proved that no new multi-frequency Craik–Criminale solutions can be formed by applying the WKB-bootstrapping algorithm.

6. A model problem

In this section, we consider a simple scalar problem which mimics some characteristics of the Navier–Stokes equations. We are interested in the real bounded field $C(x, t)$ which satisfies for $(x, t) \in (I\!R \times I\!R^+)$ the equation

$$(\partial_t + U\partial_x)C + RC = \kappa\partial_{xx}C, \tag{6.1}$$

where $U(x, t)$ and $R(x, t)$ are prescribed real functions, and $\kappa > 0$. This equation, inspired by dynamo theory (Bayly 1993), may be thought of as the advection–diffusion equation for a passive contaminant in which a stretching term has been included. Except for nonlinearity, (6.1) includes all the ingredients of the fluid flow equations (1.1).† Our main goal is here to prove that the point-wise solution obtained by FH's construction may not correspond to the local behaviour of any global solution.

FH's construction can be performed on equation (6.1) exactly as on the Navier–Stokes equations. For brevity, let us consider a single term in (1.2) such that we seek solutions to (6.1) in the form

$$C(x, t) = c(t)\mathrm{e}^{\mathrm{i}\Phi(x,t)} + \text{ complex conjugate.} \tag{6.2}$$

The phase field Φ is subject to the following constraints:

$$(\partial_t + U\partial_x)\Phi = 0, \tag{6.3a}$$

$$\Phi(x, t) = k(t)x + \delta(t), \tag{6.3b}$$

and the functions $c(t)$ and $k(t)$ are assumed to be spatially uniform. Expression (6.2) is then a solution to (6.1) in $x = \chi(t, x_0)$ if $k(t)$ and $c(t)$ satisfy

$$\mathrm{d}k/\mathrm{d}t = -[\partial_x U](\chi(t, x_0), t)k, \tag{6.4a}$$

$$\mathrm{d}c/\mathrm{d}t + R(\chi(t, x_0), t)c = -\kappa|k|^2 c. \tag{6.4b}$$

Note that the necessary conditions (4.2) for having a global solution reduce here to $c(t)\partial_x R = 0$ and $k(t)\partial_{xx}U = 0$. So, we immediately obtain that U and R must be of the form $U(x, t) = L(t)x + V(t)$ and $R(x, t) = R(t)$ to obtain a non-trivial global solution. This case is an exact analogue of the configuration studied by Craik & Criminale.

In the following, we assume that

$$U(x, t) = 2Sx, \quad R(x, t) = R_0 + R_2x^2, \tag{6.5}$$

where S, R_0 and $R_2 > 0$ are real constant parameters. For these parameters, we thus know that (6.2) is not a global solution. We shall prove that it may not be related to any global solution.

The trajectories defined by $\mathrm{d}\chi/\mathrm{d}t = U(\chi(t, x_0), t)$ with $\chi(0, x_0) = x_0$ are given by $\chi(t, x_0) = x_0\mathrm{e}^{2St}$. The stagnation point $x = 0$ is a particular trajectory to which FH's construction can be applied. It gives $k(t) = k_0\mathrm{e}^{-2St}$ and

$$c(t) = c_0 \exp\left(-R_0t + \frac{\kappa k_0^2}{4S}\left(\mathrm{e}^{-4St} - 1\right)\right), \tag{6.6}$$

with $c_0 = c(0)$ and $k_0 = k(0)$. If $S > 0$, the large-time behaviour of $c(t)$ is given by

$$c(t) \sim c_0 \exp\left(-R_0t - \frac{\kappa k_0^2}{4S}\right), \tag{6.7}$$

and therefore, $c(t)$ grows exponentially for all negative R_0.

† Nonlinearity is not essential for the present argument since the construction of FH yields identical results with or without the term $(\boldsymbol{U}' \cdot \nabla)\boldsymbol{U}'$ in (1.1).

We shall now show that no global solution exhibits such a behaviour at $x=0$. Exact global solutions of (6.1) with (6.5) may be easily constructed for any initial condition $C(x,0)=C_0(x)$ bounded at infinity. This is done by showing first that (6.1) with (6.5) possesses an infinite set of solutions of the form $C_n(x,t)=\mathrm{e}^{\lambda_n t}F_n(x)$ with

$$\lambda_n = S - R_0 - \mu(1+2n), \tag{6.8a}$$

$$F_n(x) = \mathrm{H}_n\left(\sqrt{\frac{\mu}{\kappa}}\,x\right)\exp\left(-\frac{(\mu-S)x^2}{2\kappa}\right), \tag{6.8b}$$

where n is a non negative integer, $\mu=\sqrt{S^2+R_2\kappa}$ and the functions $\mathrm{H}_n(x)$ are Hermite polynomials (Råde & Westergren 1999). Then, by using the fact that the eigenfunctions $F_n(x)$ form a complete orthogonal basis in the weighted Hilbert space with inner product

$$(F_m, F_n) = \int_{-\infty}^{+\infty} F_m(x)F_n(x)\mathrm{e}^{-Sx^2/\kappa}\mathrm{d}x = \frac{n!\,2^n\sqrt{\pi}}{\sqrt{\mu/\kappa}}\,\delta_{mn}, \tag{6.9}$$

we can expand the solution $C(x,t)$ as (Zauderer 1989):

$$C(x,t) = \sum_{n=0}^{+\infty} a_n \mathrm{e}^{\lambda_n t}F_n(x), \tag{6.10}$$

where the coefficients a_n are defined by

$$a_n = \frac{\sqrt{\mu/\kappa}}{n!\,2^n\sqrt{\pi}}(C_0, F_n) = \frac{\sqrt{\mu/\kappa}}{n!\,2^n\sqrt{\pi}}\int_{-\infty}^{+\infty} C_0(x)F_n(x)\mathrm{e}^{-Sx^2/\kappa}\,\mathrm{d}x. \tag{6.11}$$

The eigenvalues λ_n are ordered as

$$S - R_0 - \mu = \lambda_0 > \lambda_1 > \lambda_2 > \ldots \geqslant -\infty, \tag{6.12}$$

such that the large-time behaviour of $C(x,t)$ is driven by the eigenmode corresponding to the first non-zero coefficient a_n. In particular, whatever the value of $C_0(x)$, it is always bounded by an expression of the form $|C(x,t)| < b_0\mathrm{e}^{\lambda_0 t}$ for large t.

As $\lambda_0 = S - R_0 - \sqrt{S^2+R_2\kappa}$, it is possible to choose S, R_0, R_2 and κ such that $\lambda_0<0$, $S>0$ and $R_0<0$. For instance, take $S=1$, $R_0=-1$, $R_2=2$ and $\kappa=4$ which yields $\lambda_0=-1$. For such values of the parameters, all the global solutions are exponentially damped for large t for all x. By contrast, FH's solution (6.6) grows exponentially at $x=0$ for these parameters. It is therefore clear that FH's solution cannot be related to any global solution in that case.

Note that this result is not in contradiction with the short-wavelength theory of Friedlander & Vishik and Lifschitz & Hameiri for which the local behaviour at the stagnation point has been proved to be connected to a global solution. Indeed, this theory requires large k and weak diffusion such that $\kappa|k|^2=O(1)$. Therefore $\lambda_0 \sim -R_0$ when the theory applies, and a global solution can thus exhibit the same temporal behaviour as the local solution.

7. Conclusion

In this paper, we have discussed the limitations of the method used by Fabijonas & Holm to construct new solutions from a known base flow $\boldsymbol{U}$. We have shown that in general the new solution satisfies the Navier–Stokes equations in a single point of space only. We have also proved that a global solution can be formed only if the base flow satisfies important constraints. These constraints are such that no new multi-frequency Craik–Criminale solutions can be generated by the method. A simple model has also been considered for which it has been possible to show that a global solution is obtained only if the base flow has uniform velocity gradients, that is in the configuration studied by Craik & Criminale. We have further shown that when the velocity gradients are not uniform the point-wise solution obtained by the FH construction can be unphysical: it does not correspond to the local evolution of any global solution. We suspect that a similar conclusion holds for the Navier–Stokes equations. We cannot provide a mathematical proof (as it would require knowledge of all the solutions of the Navier–Stokes equations) but we can put forward the following physical argument. Consider an elliptical flow in a rotating container such as that experimentally realized by Malkus (1989) or Eloy, Le Gal & Le Dizès (2003). At the elliptic stagnation point, FH's construction reduces to the analysis by Landman & Saffman (1987) and Waleffe (1990). For small strain rate ε, Waleffe showed that there exists a solution of the form (1.2) with a constant $|k|$, which grows exponentially in time with a growth rate $\sigma=(9/16)\varepsilon-\nu|k|^2$. Because FH does not constraint $|k|$ (unlike the analysis of Lifschitz & Hameiri), this growth rate can be make positive, whatever the viscosity ν. FH's solution therefore predicts instability whatever the Reynolds number. Moreover, FH's solution grows without saturation and with the same growth rate for all time. These results are clearly unphysical and in contradiction to the experimental observations which show both the existence of a Reynolds number threshold for instability and the possible saturation of the instability mode when it develops (Malkus 1989; Eloy *et al.* 2003).

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