Stability analysis of plane wave solutions of the discrete Ginzburg-Landau equation

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The discrete Ginzburg-Landau model for a family of oscillators linearly coupled with their first neighbors is studied. The full linear stability analysis of the nonlinear plane wave solutions is performed by considering both the wave number (k) of the basic states and the wave number (q) of the perturbations as free parameters. In particular, it is shown that nonlinear plane waves can be destabilized not only by long ($q\rightarrow 0$) or short ($q=\pi$) wave perturbations, but also by intermediate wave numbers ($0<q<\pi$). Finite size effects are also considered and discussed in connection with experiments on coupled oscillating wakes.

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I. INTRODUCTION

The interest in the dynamics of discrete systems comes from the diversity of their numerous applications in physical and biological sciences. The structure of such systems implies that their dynamics is the result of an interaction between individual dynamical entities. Molecular chains and arrays of wave guides in optics [1], Josephson junctions [2] are examples referred to as coupled oscillators [3]. Also, models for life sciences can be investigated by considering the behavior of coupled cells (cardiac muscle defibrillation [4], legged locomotion [5], fireflies synchrony [6], neural networks [7]). We focus in this paper on the complex Ginzburg-Landau equation (CGLE) in one space dimension. Very often used in fluid mechanics as a phenomenological equation, the CGLE and its different forms were analytically derived in various applications (see Ref. [8], for instance) where it appears as an amplitude equation for a rapidly oscillating wave propagating in a nonlinear medium. The discretized form of the CGLE has been used for describing vortex line dynamics [9] and in the study of coupled wakes [10]. In these two examples, the oscillation of each isolated cell (each vortex or each wake) obeys a Landau equation which is the normal form of the Hopf bifurcation that gives birth to each oscillator. The global behavior of the vortex or wake arrays can consequently be described by the dynamics of coupled Hopf oscillators. Various mathematical shapes of coupling can be considered, such as linear or nonlinear, local or global [11]. Motivated by the observation of a short-range interaction between wakes [12], we restrict our analysis to a first neighbors linear coupling as it was also the case in the studies of Willaime *et al.* [9] and Le Gal [10]. Since Eckhaus [13], most stability analysis have been carried out for the continuous version of the Ginzburg-Landau equation. The instability arises from a resonance mechanism between wave trains, and is called the Benjamin-Feir instability [14] (or the sideband instability). Newell's criterion provides a condition for the destabilization of plane waves by long wavelength modulations [15]. More recently, Matkowsky and Volpert [16] considered perturbations of arbitrary wave numbers. They showed that destabilization can occur due to finite wave number perturbations which implies more complex dynamical behaviors as those usually analyzed. Direct numerical simulations of the continuous version of the CGLE also demonstrated the richness of the coupled oscillator system for which spatiotemporal chaos was in particular observed [17]. Here, it is our purpose to fully describe the stability of the nonlinear plane wave solutions of the discretized complex Ginzburg-Landau equation.

For a discrete system, the possible wave numbers of the nonlinear plane wave are related to the number of oscillators, the boundary conditions, and the nature of the coupling term between the oscillators. These wave numbers range between $k=0$ where the oscillators are in phase and $k=\pi$ where two consecutive oscillators are in phase opposition. As far as we know, the stability analysis of a nonlinear plane wave was performed for the modulation wave numbers $q\rightarrow 0$ and $q=\pi$ only [9]. In this paper, we extend this study to account for any modulation wave number.

II. THEORY

The complex Ginzburg-Landau equation takes the form

$$\frac{dA_j}{dt}=A_j-(1+ic_2)|A_j|^2A_j+\eta(1+ic_1)\times(A_{j+1}+A_{j-1}-2A_j), \tag{1}$$

where the oscillators $A_j, j\in\{1,\ldots,n\}$ also satisfy the periodicity conditions: $A_0=A_n$ and $A_{n+1}=A_1$. The real numbers c_1 and c_2 measure the influence of the coupling on the phase and its nonlinear correction, respectively. The real number η characterizes the strength of the linear coupling. This additional parameter is specific to discrete systems as for continuous systems it can be scaled out by a change of spatial variables.

The nonlinear plane wave solutions are defined by

$$A_j=Ae^{i(kj-\omega t)}, \tag{2}$$

with $i^2=-1$. The amplitude A, the wave number k, and the frequency ω satisfy the nonlinear dispersion relation

$$|A|^2=1-2\eta[1-\cos(k)]\geqslant 0, \tag{3a}$$

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$$\omega = 4\eta(c_1 - c_2)\sin^2\left(\frac{k}{2}\right) + c_2. \tag{3b}$$

For a given couple of parameters c_1 and c_2, amplitude and frequency depend on the two parameters k and $1/\eta$. The first relation defines the domain of existence of the basic state. Here it is implicitly assumed that the coupling is ''attractive'' such that the homogeneous plane wave $(k=0)$ is the only possible solution near threshold for small $1/\eta$. If the coupling was ''repulsive,'' the instability threshold would occur near $1/\eta = 4$, and the mode selection would yield the phase opposition nonlinear plane wave $(k=\pi)$ [9]. Hereafter, we focus on cases with ''attractive'' coupling. Results for ''repulsive'' coupling are deduced by the following transformations: $k \rightarrow \pi - k$ and $1/\eta \rightarrow 4 - 1/\eta$. The wave numbers k are quantified by the periodicity condition $A_{n+1} = A_1$ which implies $k_l = 2\pi l/n$ where l is an integer between 0 and the integer part of $n/2$. In the following, the analysis is done for continuous wave numbers which is equivalent to assuming that the number of oscillators is infinite. Particularities associated with a finite number of oscillators are discussed at the end of the paper in connection with numerical and experimental results.

The stability analysis is done by superimposing a perturbation $a_j(t)$ to the amplitude of the nonlinear wave at the location j:

$$A \rightarrow A + a_j(t). \tag{4}$$

The amplitude modulation is searched in a normal form:

$$a_j(t) = a e^{st} e^{iqj}, \tag{5}$$

where a is the complex constant amplitude, q the real wave number, and $s = s_r + i s_i$ the complex growth rate. In view of Eqs. (2) and (4), this amplitude modulation is in fact associated with a plane wave perturbation of wave number $k+q$. It follows that if $s_r > 0$ in Eq. (5), the underlying nonlinear plane wave of wave number k is then destabilized by a perturbation of wave number $k+q$. Below, q is used as an independent parameter for convenience, but we keep in mind that $k+q$ is the wave number of the structure that is generated by instability. Substituting expressions (2), (4), and (5) into Eq. (1) leads to a linearized equation for the perturbation and to a relation for the growth rate s which reads

$$s^2 + 2\beta s + \gamma = 0. \tag{6}$$

The real and imaginary parts of β and γ are given, respectively, by

$$\beta_r = 1 + 2\eta[-1 + C_k(1 + C_Q)],$$

$$\beta_i = 2\eta c_1 S_k S_q,$$

$$\gamma_r = 4\eta[\eta c_{11}(C_Q^2 C_k^2 - S_q^2 S_k^2) + c_{12} C_Q C_k (1 - 2\eta C_K)],$$

$$\gamma_i = 4\eta S_q S_k (c_1 - c_2)(1 - 2\eta C_K),$$

with

$$C_k = \cos(k), \quad S_k = \sin(k),$$

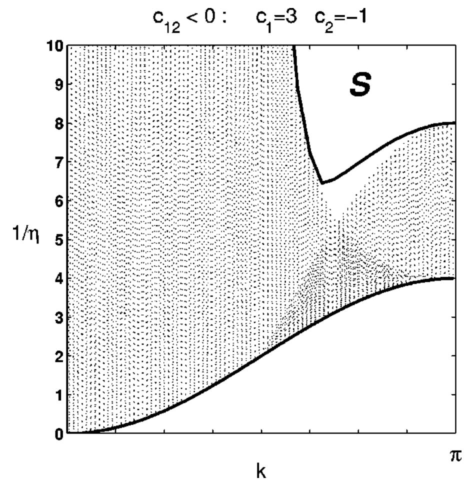

FIG. 1. Stability character of the nonlinear plane wave of wave number k versus $1/\eta$. The region S above the upper bold line is the domain of stability with respect to any modulation wave number. The dotted areas correspond to destabilization by a modulation of wave number $q \rightarrow 0$ or $q = \pi$ (see Ref. [9]). The lower bold line is the boundary of the domain of existence of the nonlinear plane waves given by condition (3a).

$$C_Q = 1 - C_q, \quad c_{11} = 1 + c_1^2, \tag{7}$$

$$C_q = \cos(q), \quad S_q = \sin(q),$$

$$C_K = 1 - C_k, \quad c_{12} = 1 + c_1 c_2.$$

After some algebraic manipulations, the condition for instability $(s_r > 0)$ reduces to

$$\beta_r < 0 \tag{8}$$

or

$$\gamma_i^2 - 4\beta_r^2\gamma_r - 4\beta_r\beta_i\gamma_i > 0 \tag{9}$$

or

$$\beta_r^2 + \beta_i^2 + \gamma_r < 0. \tag{10}$$

Note that the instability condition (8) is specific to the discretized CGLE as the analogous condition for a continuous system is never satisfied. Conditions (9) and (10) depend on the parameters c_1 and c_2, but the characteristic features mostly depend on the sign of c_{12}. If $c_{12} > 0$, conditions (9) and (10) are equivalent to the so-called Eckhaus instability condition. If $c_{12} < 0$, they are associated with the so-called Benjamin-Feir instability and here, only negative values of c_{12} are considered. Moreover, for an illustrative purpose, we shall mostly use the values $c_1 = 3$ and $c_2 = -1$ which are typical for the coupled wake arrays [19] used below for comparison. The results remain qualitatively unchanged for other values of c_1 and c_2 as long as c_{12} is negative.

III. RESULTS

Figure 1 shows the different stability zones in the $(k, 1/\eta)$ plane. The domain S above the solid line in the upper right corner corresponds to a region where nonlinear plane waves are stable with respect to modulation of any wave number q.

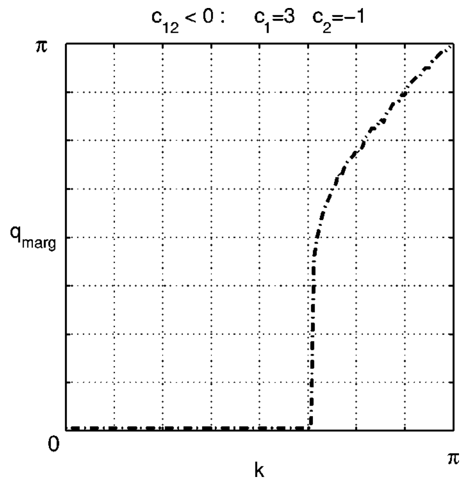

FIG. 2. Relationship between the wave numbers of the basic state (k) and of the destabilizing modulation (q) at the instability threshold.

Everywhere else above the existence curve there exists a wave number q which destabilizes the nonlinear plane wave considered. In particular, note that there is an interval in $1/\eta$ (here, $4<1/\eta<6.5$) in which no nonlinear wave is stable whatever its wave number k. Figure 1 also shows that there exists a region of the parameters where the nonlinear waves are stable with respect to the wave numbers $q\rightarrow 0$ and $q=\pi$ but unstable with respect to an intermediate wave number (between the dotted domain and the solid curve). This region of instability, located about $k=3\pi/4$, was overlooked by Willaime *et al.* [9] who focused on destabilization by the wave numbers $q\rightarrow 0$ and $q=\pi$ only. The destabilization by intermediate wave numbers is emphasized in Fig. 2 where the critical wave number q_{marg} of the destabilizing modulation on the marginal curve is plotted as a function of k. One sees that the change of stability through the wave numbers $q\rightarrow 0$ and $q=\pi$ occurs in fact only if $k\leqslant 5\pi/8$ or $k=\pi$. Information on the maximum growth rate and the corresponding modulation wave number q are displayed in Figs. 3, 4, and 5. They are obtained from a numerical investigation of Eq. (6) with continuous parameters q and k. They complete Fig. 2 by providing the most dangerous perturbation for

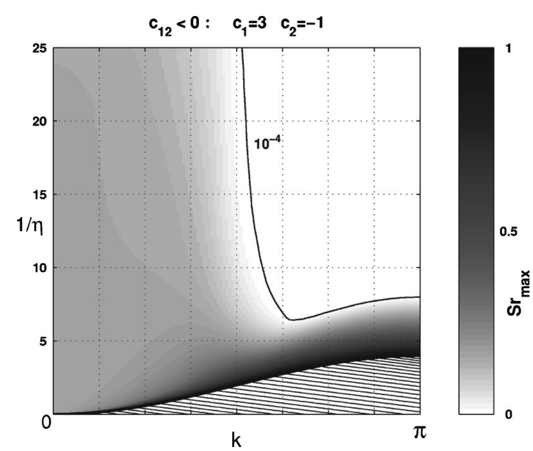

FIG. 3. Contour plot of the maximum growth rate $s_{r\max}$ in the $(k,1/\eta)$ plane. Nonlinear plane waves do not exist in the hatched region.

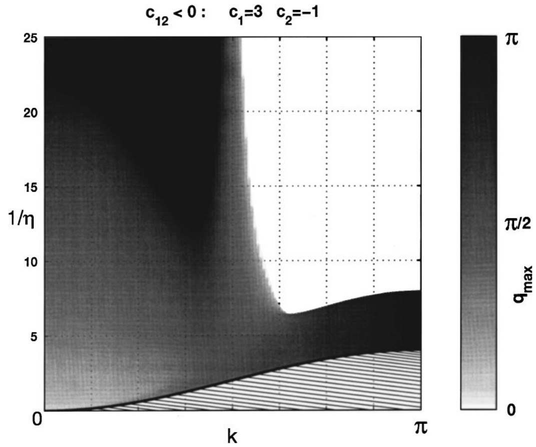

FIG. 4. Contour plot of the modulation wave number $q_{\max}$ associated with the maximum growth rate $s_{r\max}$ (see Fig. 3). In the stability domain (upper right area), nonzero wave numbers are all damped while the wave number $q=0$ is marginal. This explains the discontinuity of the contour levels across the transition curve (see also Fig. 5).

a given basic state *at any* $1/\eta$ and not only on the marginal curve. Figure 3 shows the contour plot of the maximum growth rate $s_{r\max}$ in the $(k,1/\eta)$ plane. The region of stability (region S in Fig. 1) is recovered in the top-right corner. In Fig. 4, contour levels of the most unstable wave number $q_{\max}$ are given in the $(k,1/\eta)$ plane. Figure 5 displays cuts of both previous contour plots for different values of the wave number k. These figures show interesting features of the instability. First, they demonstrate that the largest growth rates are reached close to the existence curve and in general for a modulation wave number different from $q\rightarrow 0$ and $q=\pi$. Again, this clearly indicates the limitation of any stability analysis based on these two limits only. They also show that, for large $1/\eta$, all the basic states with $k<\pi/2$ are preferentially destabilized by the wave number $q=\pi$. Moreover, the

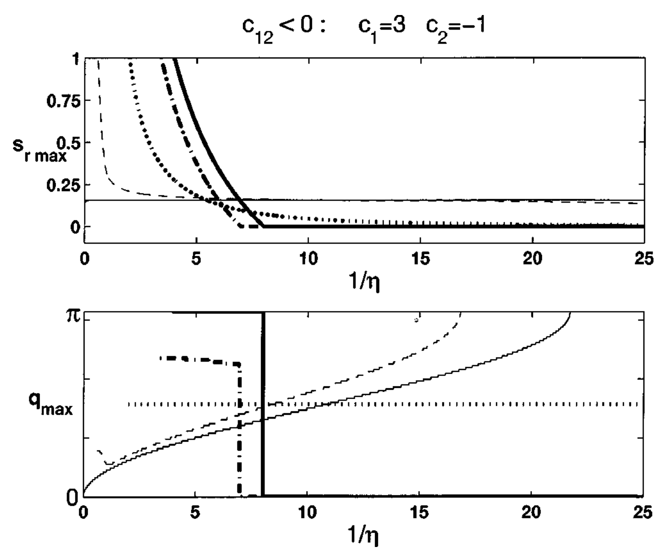

FIG. 5. Variations of $s_{r\max}$ (top) and $q_{\max}$ (bottom) as a function of $1/\eta$ for different basic state wave numbers ($k=0$: —, $k=\pi/4$: − −, $k=\pi/2$: ⋯, $k=3\pi/4$: − · −, and $k=\pi$: −). Top and bottom figures correspond to vertical cuts in Figs. 3 and 4, respectively.

growth rate associated with this instability is relatively small and almost independent of k.

IV. DISCUSSION

As explained above the growth of the modulation q on the nonlinear wave k implies the generation of another plane wave of wave number $k+q$. The nonlinear competition between this emerging wave and the underlying nonlinear wave is a difficult issue which is not addressed here. However, the knowledge of the stability properties of the nonlinear plane wave for any $1/\eta$ is useful to infer the dynamical evolution of the system as $1/\eta$ is progressively increased. In particular, for small $1/\eta$ one expects the system to be preferentially in phase, because plane waves with small wave number are the only existing states. This in-phase behavior should be even more pronounced as the number n of oscillators is smaller. Indeed, the threshold $1/\eta_o = 2[1 - \cos(2\pi/n)]$ below which the nonlinear plane wave $k=0$ is the only existing wave, increases as n decreases. Moreover, as soon as this state is established, Fig. 5 (bottom) shows that it is first destabilized by the smallest nonzero wave number that is $q=2\pi/n$. An explicit expression for the critical parameter $1/\eta_c$ at which the destabilization occurs can be obtained from the instability condition [Eqs. (8)–(10)]. After simplification, it reads

$$\frac{1}{\eta_c} = -\frac{c_{11}}{c_{12}}[1-\cos(2\pi/n)], \tag{11}$$

where the parameters c_{11} and $c_{12}<0$ have been defined in Eq. (7). For the values of c_1 and c_2 taken above, $c_{11}/c_{12} = -5$, thus $1/\eta_c = \frac{5}{2} 1/\eta_o$. Expression (11) shows that the domain of stability of the solution with the wave number $k=0$ grows when n decreases. Note, however, that the size of this domain depends on c_1 and c_2: the closer c_{12} is from zero, the larger the domain of stability. When $1/\eta > 1/\eta_c$, the wave number $k=2\pi/n$ is expected to appear. It may become dominant and therefore be itself destabilized by another modulation along a similar process, and so on. As $1/\eta$ is increased in an intermediate range (here $1<1/\eta<8$), more nonlinear waves are available and each of them becomes unstable with respect to several modulation wave numbers. In this regime, a strongly disordered spatiotemporal evolution is then highly probable. For large $1/\eta (1/\eta>8)$, all nonlinear plane waves are possible but small wave number states are unstable while large wave number (close to π) states are stable. A small wave number state is then expected to evolve to a larger wave number state by the growing of instability modes until it reaches a stable configuration (in the upper right region above the bold line in Fig. 3) with a wave number larger than $\pi/2$. Note also that for even larger $1/\eta$ $(1/\eta>22)$, the unstable in-phase basic state $(k=0)$ should immediately evolve to a stable phase-opposition state $(k=\pi)$ as the most unstable wave number is $q=\pi$ in such a case.

V. CONCLUSION

These predictions were successfully checked by direct numerical simulations of the discretized CGLE. We observed indeed that the system of oscillators evolves from an in-phase state to a phase-opposition state with an intermediate disordered state when $1/\eta$ is progressively increased, i.e., when the coupling between the oscillators is progressively weakened. They are also in agreement with experimental results on the regimes of the flow behind rows of cylinders [12,18,19]. In that case, the wake behind each cylinder, that is the Bénard–von Karman vortex street, represents a single oscillator which is strongly coupled to its neighbors if the cylinders are close to each other and weakly coupled if they are far apart. For the two situations (strong coupling and weak coupling), the wakes were found in phase and in opposition of phase, respectively, as predicted by our analysis. Strongly disordered states with nucleation of defects were also observed in the experiment for intermediate coupling [19]. Note that the wave number $q=0$ is neutral whatever k, so one has to consider the limit $q\rightarrow 0$ to analyze the stability of a nonlinear plane wave with respect to long-wavelength modulations.

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