Three-Dimensional Temporal Spectrum of Stretched Vortices

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The three-dimensional stability problem of a stretched stationary vortex is addressed in this Letter. More specifically, we prove that the discrete part of the temporal spectrum is associated only with two-dimensional perturbations. [S0031-9007(97)02854-8]

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Numerical simulations [1–3] as well as real experiments [4] indicate that vorticity in turbulent flows concentrates in localized regions such as filaments which are fairly well described by stretched vortices such as the celebrated Burgers vortex solution [5] or Moffatt, Kida, and Ohkitani's asymptotic solution [6]. If one agrees that these local structures are important dynamical objects of the global turbulent field, their temporal stability with respect to generic perturbations should be addressed. So far, this problem has been studied only for Burgers' vortex and, in that case, for purely two-dimensional perturbations. Robinson and Saffman [7] provided an analytical solution for low Reynolds numbers, and their results were later numerically extended by Prochazka and Pullin [8] up to Reynolds numbers $\mathrm{Re} = 10^4$. These papers on Burgers vortex indicate that the temporal spectrum associated with two-dimensional perturbations is discrete and corresponds to damped modes. On the contrary, the general stability problem of stretched vortices has not been tackled yet. Except for the 2D stability analysis of axisymmetric Burgers vortex, it does not reduce to a classical eigenvalue problem with a single ODE to solve. Indeed, infinitesimal 3D perturbations are affected by the presence of stretching along the vortex axis which precludes the reduction of the problem by Fourier analysis.

In this paper, we prove that the discrete part of the temporal spectrum is only associated with two-dimensional perturbations. In Sec. (I), the stability problem is introduced and particular time-dependent solutions are exhibited. Their existence imposes conditions on the 3D temporal mode structure. In Sec. (II), these conditions are shown to be consistent only for modes independent of the vortex axis coordinate.

(I) *Modified Fourier decomposition.*—Let us consider a stationary velocity field $\mathbf{U}_0 = (U_0, V_0, W_0)$ of the form

$$U_0 = \frac{\partial \phi}{\partial x}(x,y) + U_v(x,y), \tag{1a}$$

$$V_0 = \frac{\partial \phi}{\partial y}(x,y) + V_v(x,y), \tag{1b}$$

$$W_0 = \gamma z, \tag{1c}$$

where (U_v, V_v) and $(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y), \gamma z)$, respectively, stand for a localized rotational field and a global velocity field satisfying $\nabla^2\phi = -\gamma$. Such an expression represents a stationary stretched vortex aligned with the z axis and subjected to a global strain field. In the sequel, the strain rate γ along the vortex axis is assumed to be positive.

The structure of such a solution is governed by the balance between stretching due to the global strain field and viscous diffusion. In particular, the core size should scale as $\sqrt{\nu/\gamma}$ where ν is the kinematic viscosity. In the simplest case of an axisymmetric strain $(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y), \gamma z) = (-\gamma x/2, -\gamma y/2, \gamma z)$, one recovers the Burgers solution. Other examples are Robinson and Saffman [7] and Moffatt, Kida, and Ohkitani [6] solutions which correspond to the nonaxisymmetric case at small and large Reynolds numbers, respectively. In the subsequent analysis, expression (1a)–(1c) is considered as the basic flow where U_v, V_v and ϕ are not specified.

The dynamics of pressure and velocity infinitesimal perturbations $(\mathbf{u}, p)$ around (1a)–(1c) is described by the linear system

$$\partial_t \mathbf{u} + \mathbf{U}_0 \cdot \nabla \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U}_0 = -\nabla p + \nabla^2 \mathbf{u}, \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2b}$$

The above system being homogeneous with respect to time, one might look at the temporal spectrum of such a system. Modes belonging to the discrete part of this spectrum read

$$(\mathbf{u}_\omega, p_\omega) = (\mathbf{v}_\omega(x,y,z), q_\omega(x,y,z))e^{-i\omega t}. \tag{3}$$

Inserting (3) into (2a), (2b), one obtains equations for $\mathbf{v}_\omega(x,y,z)$ and $q_\omega(x,y,z)$ which are nonseparable with respect to any spatial variable. In such a case, the use of standard Fourier analysis does not simplify any further the problem. However, the z dependence in Eq. (2a) appears only through the uniform strain along the z axis, i.e., through the term $\gamma z \partial_z$. Such a simple dependence allows one to search for time-dependent solutions which are different from (3). These are "generalized Fourier modes" in the z direction with a time-dependent wave number

$$(\mathbf{u}, p) = (\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0))e^{ik(t)z}e^{-\nu\int_0^t [k(s)]^2 ds}, \tag{4}$$

where the initial wave number condition $k_0 = k(0)$ is a free parameter. Indeed, as soon as the time evolution of $k(t)$ is appropriately chosen, more precisely if $k(t) = k_0 e^{-\gamma t}$, the nonhomogeneous term in z is removed in (2a). The system (2a), (2b) is then reduced to a couple of equations homogeneous in z, which describe the time evolution of the modified Fourier modes components $\check{u}_x$ and $\check{u}_y$:

$$(\mathcal{L} + \partial_x U_0)\check{u}_x + \partial_y U_0 \check{u}_y = \frac{e^{2\gamma t}}{k_0^2}\frac{\partial}{\partial x}(\mathcal{L} + 2\gamma) \times \left(\frac{\partial \check{u}_x}{\partial x} + \frac{\partial \check{u}_y}{\partial y}\right), \tag{5a}$$

$$(\mathcal{L} + \partial_y V_0)\check{u}_y + \partial_x V_0 \check{u}_x = \frac{e^{2\gamma t}}{k_0^2}\frac{\partial}{\partial x}(\mathcal{L} + 2\gamma) \times \left(\frac{\partial \check{u}_x}{\partial x} + \frac{\partial \check{u}_y}{\partial y}\right), \tag{5b}$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x} + V_0\frac{\partial}{\partial y} - \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{6}$$

Finally, note that $\check{u}_z$ is given by the components $\check{u}_x$ and $\check{u}_y$ through the continuity equation.

The temporal mode (3) may be decomposed upon a basis of such modified Fourier modes (4). Let us expand the spatial part $(\mathbf{v}_\omega, q_\omega)$ of (3) in the usual Fourier modes along the z direction:

$$(\mathbf{v}_\omega, q_\omega) = \int_{-\infty}^{+\infty} (\mathbf{v}(x,y,k_0), q(x,y,k_0))e^{ik_0 z} dk_0 . \tag{7}$$

This expansion can be viewed as a superposition at time $t = 0$ of generalized Fourier modes provided the following initial condition is satisfied:

$$(\check{\mathbf{u}}(x,y,0,k_0), \check{p}(x,y,0,k_0)) = (\mathbf{v}(x,y,k_0), p(x,y,k_0)) . \tag{8}$$

According to (4), each generalized Fourier mode evolves independently: An alternative expression for the temporal mode (3) is thus provided for all t

$$(\mathbf{u}_\omega, p_\omega) = \int_{-\infty}^{+\infty} (\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0)) \times e^{ik_0 e^{-\gamma t} z} e^{-\nu k_0^2 (1-e^{-2\gamma t})/2\gamma} dk_0 . \tag{9}$$

The above expression is consistent with (3) if the following equality holds at any x and y locations, time t and wave number k_0:

$$(\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0))e^{-\nu k_0^2(1-e^{-2\gamma t})/2\gamma} = (\mathbf{v}(x,y,k_0 e^{-\gamma t}), q(x,y,k_0 e^{-\gamma t}))e^{-\gamma t}e^{-i\omega t} . \tag{10}$$

In the following section, equality (10) is shown to be valid only for $k_0 = 0$: Discrete temporal modes are bound to be two dimensional.

(II) *The Z dependence of a three-dimensional temporal mode.*—First a cutoff wave number k_c above which $\mathbf{v}(x,y,k_0)$ and $q(x,y,k_0)$ vanish, should exist. Indeed, assume that such a cutoff does not appear, large wave numbers are then present in the spatial spectrum of (3). It is thus possible to take the simultaneous limits $k_0 t$ large and t small in (10). The right-hand side of this equation becomes

$$(\mathbf{v}(x,y,k_0 e^{-\gamma t}), q(x,y,k_0 e^{-\gamma t}))e^{-\gamma t}e^{-i\omega t} \sim (\mathbf{v}(x,y,k_0), q(x,y,k_0)) . \tag{11}$$

In order to estimate the left-hand side of (10), the behavior of $(\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0))$ is evaluated using Eqs. (5a), (5b). Two cases are to be considered according to the characteristic spatial variations of $\check{u}_x$ and $\check{u}_y$ in the x and y directions. When, for large axial wave number k_0, these components evolve over spatial scales independent of k_0, the right-hand side of (5a), (5b) can be neglected and the leading order time evolution is independent on k_0. This means that, for $k_0 t$ large and t small, the left-hand side of (10) reads

$$(\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0))e^{-\nu k_0^2(1-e^{-2\gamma t})/2\gamma} \sim (\check{\mathbf{u}}_\infty(x,y,0), \check{p}_\infty(x,y,0))e^{-\nu k_0^2 t} . \tag{12}$$

On the contrary, when $\check{u}_x$, $\check{u}_y$ evolve over spatial scales comparable to $1/k_0$, the correct expression is found when introducing in (5a), (5b) new fast variables

$$\overline{x} = k_0 x; \qquad \overline{y} = k_0 y; \qquad \overline{t} = k_0^2 t , \tag{13}$$

in addition to x and y, and thereafter expanding the solution with respect to $1/k_0$. At leading order, homogeneous equations are obtained:

$$\frac{\partial \check{u}_x}{\partial \overline{t}} - \nu\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2}\right)\check{u}_x = e^{2\gamma t}\left[\frac{\partial}{\partial \overline{t}} - \nu\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2}\right)\right] \times \frac{\partial}{\partial \overline{x}}\left(\frac{\partial \check{u}_x}{\partial \overline{x}} + \frac{\partial \check{u}_x}{\partial \overline{y}}\right), \tag{14a}$$

$$\frac{\partial \check{u}_x}{\partial \overline{t}} - \nu\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2}\right)\check{u}_y = e^{2\gamma t}\left[\frac{\partial}{\partial \overline{t}} - \nu\left(\frac{\partial^2}{\partial \overline{x}^2} + \frac{\partial^2}{\partial \overline{y}^2}\right)\right] \times \frac{\partial}{\partial \overline{x}}\left(\frac{\partial \check{u}_x}{\partial \overline{x}} + \frac{\partial \check{u}_x}{\partial \overline{y}}\right). \tag{14b}$$

For t small, the term $e^{2\gamma t}$ may be taken equal to 1 and the general solution of (14a) be written in the fast variables $\overline{x}$ and $\overline{y}$ via the usual Fourier decomposition

$$(\check{u}_x, \check{u}_y) = \int (\check{u}_x^{(0)}(k_x, k_y, x, y), \check{u}_y^{(0)}(k_x, k_y, x, y)) e^{-\nu(k_x^2+k_y^2)\overline{t}} e^{ik_x\overline{x}+ik_y\overline{y}}\, dk_x\, dk_y\,. \quad (15)$$

For fixed x and y and $k_0 t \rightarrow +\infty$, the above expression (15) is evaluated by the steepest descent method

$$(\check{u}_x, \check{u}_y) \sim \frac{4\pi}{\nu k_0^2 t} (\check{u}_x^{(0)}(0,0,x,y), \check{u}_y^{(0)}(0,0,x,y))\,. \quad (16)$$

Similar behaviors can be obtained for $\check{u}_z$ and $\check{p}$. For this case, the right-hand side of (10) now reads

$$(\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0)) e^{-\nu k_0^2(1-e^{-2\gamma t})/2\gamma} \sim (\check{\mathbf{u}}^{(0)}(0,0,x,y), \check{p}^{(0)}(0,0,x,y)) \frac{4\pi}{\nu k_0^2 t} e^{-\nu k_0^2 t}\,. \quad (17)$$

Introducing into Eq. (10) estimates (11) and (12) or (17) leads to an inconsistency for large wave numbers k_0. This contradiction implies that the spatial spectrum of (3) vanishes for sufficiently large wave numbers: There hence exists a cutoff wave number k_c such that $(\mathbf{v}(x,y,k_0), q(x,y,k_0)) = 0$ for $k_0 > k_c$.

Consider now a wave number $0 < k_1 < k_c$ such that $(\mathbf{v}(x,y,k_1), q(x,y,k_1)) \neq (\mathbf{0}, 0)$ and a time t_1 such that $k_0 = k_1 e^{\gamma t_1} > k_c$. Equation (10) then implies that $(\check{\mathbf{u}}(x,y,t_1,k_0), \check{p}(x,y,t_1,k_0)) \neq 0$, which leads again to a contradiction. Indeed, the initial condition $(\check{\mathbf{u}}(x,y,0,k_0), \check{p}(x,y,0,k_0))$ is equal to zero since $k_0 > k_c$, the quantity $(\check{\mathbf{u}}(x,y,t,k_0), \check{p}(x,y,t,k_0))$ thus remains zero for all $t > 0$ since it is governed by Eqs. (5a), (5b). This imposes that $(\mathbf{v}(x,y,k_1), q(x,y,k_1)) = (\mathbf{0}, 0)$ for all $k_1 \neq 0$. Discrete temporal modes are then independent upon z, i.e., purely two-dimensional ones. For the axisymmetric Burgers vortex, those have been computed by Robinson and Saffman [7] and Prochazka and Pullin [8]. These authors showed that these modes are damped.

Needless to say, the general stability problem for Burgers vortex or any stretched vortex cannot be solved by looking only at the discrete part of the temporal spectrum. In particular, the analysis of the continuous spectrum needs to be considered. However, the process of "bidimensionalization" displayed in (4) can be used again in that case.

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