On the three-dimensional instability of elliptical vortex subjected to stretching

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It is known that two-dimensional vortices are subject to generic three-dimensional instabilities. This phenomenon is located near the core of vortices and depends on the eccentricity of their streamlines. In this paper we are concerned with the modification of this instability when stretching is applied to such vortices. We describe this instability by linearizing the Navier–Stokes equations around a basic state, which is an exact time-dependent solution. The complete system for the perturbations is reduced to a single equation for the perturbed velocity along the vortex span. In the limit of weak stretching, a perturbation theory can be performed and leads to a WKBJ approximation for the solution. This procedure demonstrates that a small amount of stretching is able to prevent the appearance of three-dimensional instabilities for vortices with a low enough eccentricity. Since most vortices are slightly elliptical in turbulent flows, the above computations are expected to cover a wide range of experimental cases. In particular, it is tentatively argued that this mechanism may explain recent experimental observations [Phys. Fluids **7**, 630 (1995)]. © *1996 American Institute of Physics.* [S1070-6631(96)00208-5]

I. INTRODUCTION

The presence of organized structures in turbulent flows has been recently emphasized by physical experiments[1] and numerical simulations.[2–4] In particular, vorticity is mainly concentrated in localized tube-like regions: the so-called ''worms'' [3] or ''sinews.'' [5] If one agrees that such local structures are important for the global turbulent field, it is certainly worth studying their elementary dynamics. In this spirit, Moffatt, Kida, and Ohkitani[5] studied the effect of an asymmetric strain on a Burgers vortex to understand the influence of the average flow field on high vorticity regions. They showed, in particular, that vortices become slightly elliptically shaped when subjected to an asymmetric strain field.

The elliptical character of vortices has also been the focus of much work during the last decade. Indeed, a universal three-dimensional inviscid short-wave instability[6] connects two features of inhomogeneous turbulent flows: the presence of two-dimensional vortices and three-dimensional small scales. This purely inviscid phenomenon is due to the elliptical nature of the two-dimensional vortex streamlines, as explained by the linear perturbation approaches of Bayly,[7] Waleffe,[8] and Landman and Saffman.[9] Their analysis which is locally valid near the center of the vortex, has been generalized by Lifschitz and Hameiri,[10] who perform a WKBJ analysis along any vortex streamline. A generic feature of this instability is the production of small scales directly from a smooth basic state.

Taking into account these two separate standpoints, it is natural to ask how the elliptical instability behaves when a velocity component is added along the span of the previously two-dimensional vortex. This is not a purely academic question since vortices present in turbulent flows are not purely two dimensional but experience a *three-dimensional* strain arising from the mean field. In this work, we specifically consider the stability of the core of a three-dimensional elliptical vortex subjected to an axial strain. An analogous study was performed in Craik and Allen,[11] where the influence on the elliptical instability of a periodic strain was considered. Note that stratification and Coriolis effects may also alter this three-dimensional instability.[12–14]

When stretching is present, no steady solution such as the two-dimensional elliptic vortex is known for the purely inviscid case. Two cases are possible: the vortex is time-dependent, the stretching effect increasing the vorticity level with time or viscous effects are strong enough to counterbalance this process. This latter possibility actually means that vortices are long-lived enough to be structurally dependent on viscosity. An example is the vortex studied in Moffatt *et al.*[5] In this paper we disregard that case which will be addressed in future work and consider only the time-dependent vortex for which vorticity is enhanced by stretching. For this case, an exact solution of the Navier–Stokes equations can be found, that represents the vortex core. The linear stability of such a state is performed in this paper, which is organized as follows. In Sec. II, the basic stretched

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vortex solution is provided. Thereafter the equations for the velocity perturbations are established and reduced to a single equation, which is unfortunately nonseparable in space and time variables. However, it is possible to find an equivalent of the usual Fourier modes decomposition. This is done in Sec. III by searching perturbations in the form of ''inertial waves.'' Time-dependent equations for both amplitude and wave number are thus obtained. Finally, Sec. IV provides a heuristic description of the dynamics of these equations, which is put on firmer grounds in Sec. V using WKBJ approximations.

II. BASIC FORMULATION

Consider a three-dimensional vortex subjected to stretching along its span, which is produced by the mean flow of surrounding vortices. This strain field is supposed to take the local form $U_s=(\alpha x,\beta y,\gamma z)$, where $\alpha+\beta+\gamma=0$ to comply with the incompressibility condition. Moreover, we focus on the case of an axial strain for which coefficient γ is positive while α and β are both negative. In such a case, the vortex tube aligns with the z axis, the stretching direction. The total velocity then reads as

$$\mathbf{U}_0=\begin{pmatrix} U_0 \\ V_0 \\ W_0 \end{pmatrix}=\begin{pmatrix} \alpha x+\partial_y\Psi \\ \beta y-\partial_x\Psi \\ \gamma z \end{pmatrix}, \tag{1}$$

where $\Psi(x,y,t)$ stands for the induction incoming from the vortex tube itself. When viscous diffusion controls the structure of the solution, there exists a steady vortex the core of which scales with $\sqrt{\nu/\gamma}$. This solution is nothing but the Burgers vortex for axisymmetric strain or the vortex analyzed by Moffatt *et al.*[5] for the more general nonaxisymmetric case.

By contrast, for larger vortices, no steady solution exists since viscosity is then unable to balance the stretching effect, which tends to concentrate vorticity. However, it is possible to construct an exact time-dependent solution of the Navier–Stokes equations that appropriately describes this evolution. Suppose that $\Psi(x,y,t)=-[\mu(t)/2](x^2+y^2)$. In that case, the basic flow given by (1) is the superposition of the uniform strain and a time-dependent uniform vorticity field $(0,0,2\mu)$. As shown by Craik,[12] this expression is an exact solution of the Navier–Stokes equations, provided that the vorticity evolves according to the law

$$\mu(t)=\mu_0 e^{\gamma t}. \tag{2}$$

If a $\pi/4$ rotation of the coordinates with respect to the z axis is performed, this basic flow U_0 takes the following compact form:

$$\mathbf{U}_0=\begin{pmatrix} -\gamma/2 & -\mu(t)+\epsilon & 0 \\ \mu(t)+\epsilon & -\gamma/2 & 0 \\ 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{3}$$

where $\epsilon\equiv(\alpha-\beta)/2$ is a measure of the strain asymmetry and is related to the streamlines eccentricity. (In order for the streamlines to be elliptic, one assumes that $\mu_0>\epsilon$.) When stretching γ vanishes, the two-dimensional elliptical flow studied by Pierrehumbert[6] and Bayly,[7] among others, is recovered. Note that this solution is more general than it appears since it may represent the local approximation of a generic evolving vortex near its core. In the subsequent analysis, expression (3) is considered as the basic flow.

The dynamics of pressure and velocity perturbations $(p,\mathbf{u})$ around this basic state is described by the following system:

$$D_t\mathbf{u}+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{U}_0+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=-\boldsymbol{\nabla}p+\nu\,\nabla^2\mathbf{u}, \tag{4a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0, \tag{4b}$$

where the convective time derivative $D_t=\partial_t+\mathbf{U}_0\cdot\boldsymbol{\nabla}$ explicitly depends on spatial variables through the basic state.

Suppose one can neglect the nonlinear term $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ in (4a); a straightforward algebraic manipulation of (4a) and (4b) then leads to the following equations:

$$\nabla^2 p=2\mu(\partial_x v-\partial_y u)-2\epsilon(\partial_x v+\partial_y u)-3\gamma\,\partial_z w, \tag{5a}$$

$$(D_t-\gamma-\nu\nabla^2)(\partial_x v-\partial_y u)=2\mu\,\partial_z w, \tag{5b}$$

$$(D_t-\gamma-\nu\nabla^2)(\partial_x v+\partial_y u)=2\epsilon\,\partial_z w-2\partial_x\,\partial_y p, \tag{5c}$$

$$(D_t+\gamma-\nu\nabla^2)w=-\partial_z p. \tag{5d}$$

It is possible to reduce the above system to a single equation for the perturbation velocity w along the z axis. First, one differentiates (5a) with respect to $\partial_z(D_t-\gamma-\nu\nabla^2)$. Noting that this latter operator is equal to $(D_t-\nu\nabla^2)\partial_z$, and using (5b)–(5d), one gets

$$\begin{aligned}&[D_t-\nu\nabla^2]\nabla^2[D_t+\gamma-\nu\nabla^2]w+4(\mu^2-\epsilon^2)\partial_z^2 w\\&\quad-4\epsilon\,\partial_x\partial_y[D_t+\gamma-\nu\nabla^2]w-3\gamma[D_t-\nu\nabla^2]\partial_z^2 w\\&\quad=-2\,\partial_t\mu\,\partial_z(\partial_x v-\partial_y u).\end{aligned} \tag{6}$$

Furthermore, a second relationship between w and $(\partial_x v-\partial_y u)$ is obtained using Eqs. (5b) and (5d) and the continuity equation (4b),

$$\begin{aligned}&\nabla_\perp^2\nabla^2(D_t+\gamma-\nu\nabla^2)w\\&\quad=-[2\mu\nabla_\perp^2-2\epsilon(\partial_x^2-\partial_y^2)]\partial_z(\partial_x v-\partial_y u)\\&\qquad-[4\epsilon\partial_x\partial_y\partial_\epsilon^2-3\gamma\,\partial_z^2\nabla_\perp^2]w,\end{aligned} \tag{7}$$

with $\nabla_\perp^2=\partial_x^2+\partial_y^2$. Finally, if one accounts for relations (6) and (7), one gets

$$\begin{aligned}&[2\mu\nabla_\perp^2-2\epsilon(\partial_x^2-\partial_y^2)]\{[D_t-\nu\nabla^2]\nabla^2[D_t+\gamma-\nu\nabla^2]w\\&\quad+4(\mu^2-\epsilon^2)\partial_z^2 w-4\epsilon\,\partial_x\partial_y[D_t+\gamma-\nu\nabla^2]w\\&\quad-3\gamma[D_t-\nu\nabla^2]\partial_z^2 w\}\\&\quad=2\gamma\mu\{\nabla_\perp^2\nabla^2(D_t+\gamma-\nu\nabla^2)w\\&\qquad+[4\epsilon\partial_x\partial_y\partial_z^2-3\gamma\,\partial_z^2\nabla_\perp^2]w\}.\end{aligned} \tag{8}$$

Note that Eq. (8) is nonhomogeneous in space and time variables. This implies that the standard Fourier analysis is of no use and one must resort to a new method to solve the problem. Fortunately enough, one is guided by the specific case $\gamma=\epsilon=0$, which corresponds to a solid body rotation. It is known that this flow supports neutral waves called inertial waves for which analytical expressions are available. In the following section, our purpose is to construct the equivalent of these solutions for the general case $\gamma\neq 0$ and $\epsilon\neq 0$.

III. INERTIAL WAVE PERTURBATIONS

As in the case of solid body rotation, we look for perturbations in the form of plane waves with time-dependent wave vectors (sometimes called Kelvin waves):

$$(\mathbf{u},p)=[\tilde{u}(t),\tilde{v}(t),\tilde{w}(t),\tilde{p}(t)]e^{i\mathbf{k}(t)\cdot\mathbf{x}}. \tag{9}$$

If the time evolution of the wave vector $\mathbf{k}(t)$ is appropriately chosen, the nonhomogeneous terms in Eqs. (4a) and (4b) are removed. Similar to what happens when the usual Fourier expansion is applied, Eq. (8) then reduces to an ordinary differential equation. It is readily found that this term cancellation occurs if the wave vector $\mathbf{k}(t)$ satisfies

$$D_t(e^{i\mathbf{k}(t)\cdot\mathbf{x}})=0, \tag{10}$$

This equation, which is interpreted as a phase conservation for plane wave solutions, also reads as a system of three scalar equations:

$$\partial_t k_x=\frac{\gamma}{2}k_x-(\mu_0 e^{\gamma t}+\epsilon)k_y, \tag{11a}$$

$$\partial_t k_y=(\mu_0 e^{\gamma t}-\epsilon)k_x+\frac{\gamma}{2}k_y, \tag{11b}$$

$$\partial_t k_z=-\gamma k_z. \tag{11c}$$

When γ is nonzero, Eq. (11c) shows that the wave number $k_z=k_0^{\parallel}e^{-\gamma t}$ tends to zero. Moreover, Eqs. (11a) and (11b) can be analytically solved for two specific instances. The first one corresponds to $\gamma=0$. In that case, k_z is a constant and the wave vector periodically precesses around the z axis following an ellipse:[8]

$$k_x=\frac{k_0^{\perp}}{\sqrt{E}}\cos[\sqrt{\mu^2-\epsilon^2}t+\xi_0], \tag{12a}$$

$$k_y==k_0^{\perp}\sqrt{E}\,\sin[\sqrt{\mu^2-\epsilon^2}t+\xi_0], \tag{12b}$$

where E is the ellipticity $E\equiv\sqrt{(\mu-\epsilon)/(\mu+\epsilon)}$. When eccentricity $\epsilon=0$, a general solution can be found as well:

$$k_x=k_0^{\perp}e^{\gamma t/2}\cos[\mu_0 e^{\gamma t}+\xi_0], \tag{13a}$$

$$k_y=k_0^{\perp}e^{\gamma t/2}\sin[\mu_0 e^{\gamma t}+\xi_0], \tag{13b}$$

which shows that the wave vector amplitude and its angle with the z axis increase while it rotates more and more rapidly around the same axis. In the next section, we provide an approximation for the general solution to (11a) and (11c), which shares the properties of the above cases.

Let us now write the ordinary differential equation for the wave amplitude. Note first that viscosity always appears in the system (5a)–(5d) through the operator $(D_t-\nu\nabla^2)$. Therefore, viscous diffusion acts on the plane wave (9) as

$$[\tilde{u}(t),\tilde{v}(t),\tilde{w}(t),\tilde{p}(t)]=[\tilde{U}(t),\tilde{V}(t),\tilde{W}(t),\tilde{P}(t)]\exp\left(-\nu\int_0^t|k(s)|^2\,ds\right), \tag{14}$$

where $\tilde{\mathbf{U}}$ and $\tilde{P}$ are independent of viscosity. Using Eqs. (10) and (14), one easily determines the following time-dependent equation for $\tilde{W}$:

$$\partial_t^2(|k|^2\tilde{W})+[4\mu^2k_z^2-4\epsilon^2|k|^2-4\epsilon\mu(k_y^2-k_x^2)]\tilde{W}=2\gamma\mu\,\frac{|k_\perp|^2\partial_t(|k|^2\tilde{W})+4\epsilon k_xk_y|k|^2\tilde{W}}{2\mu|k_\perp|^2+2\epsilon(k_y^2-k_x^2)}. \tag{15}$$

Note that $2\mu|k_\perp|^2+2\epsilon(k_y^2-k_x^2)$ cannot vanish since μ is always larger than ϵ and $|k_\perp|^2=k_x^2+k_y^2$. Together with (11a)–(11c), this second-order equation describes the evolution of the generalized inertial waves (9),(14).

It is worth mentioning that the continuity equation guarantees that such plane waves are transverse, i.e. $\mathbf{U}\cdot\mathbf{k}=0$. As a byproduct, these linear solutions actually satisfy the full nonlinear Navier–Stokes equations since $(\mathbf{u}\cdot\nabla)\mathbf{u}=i(\mathbf{u}\cdot\mathbf{k})\mathbf{u}$ is identically zero for each of them.

Although the above equations were obtained for any stretching parameter γ, we shall now restrict our investigation to the case $\gamma\ll\mu_0$. Indeed, when the stretching is weak, the axial vorticity $2\mu(t)$ grows slowly and the basic state (3) is physically relevant for a sufficiently long time to allow a temporal stability analysis. When $\gamma=O(\mu_0)$, such a stability analysis is irrelevant since, in contrast with the basic flow given by (3), the core of the vortex will be altered by viscous effects on the reference time scale $1/\mu_0$.

IV. HEURISTIC APPROACH

Equations (11a)–(11c) are nondimensionalized using as reference length scale $1/k_0^{\parallel}$ and reference time scale $1/\mu_0$, where $2\mu_0$ is the initial vorticity. This introduces the dimensionless quantities $\gamma^*=\gamma/\mu_0$, $\epsilon^*=\epsilon/\mu_0$. In the sequel, dimensionless equations and quantities are always considered and asterisks are dropped.

When γ is small, the wave vector equations can be integrated by the WKBJ method,[15] which leads to the following approximations:

$$k_x=k_0^{\perp}\frac{e^{T/2}}{\sqrt{E(T)}}\cos\left(\frac{1}{\gamma}\int_0^T\Omega(s)ds+\xi_0\right)+O(\gamma\epsilon,\gamma^2), \tag{16a}$$

$$k_y=k_0^{\perp}\sqrt{E(T)}e^{T/2}\sin\left(\frac{1}{\gamma}\int_0^T\Omega(s)ds+\xi_0\right)+O(\gamma\epsilon,\gamma^2), \tag{16b}$$

$$k_z=e^{-T}, \tag{16c}$$

where the ellipticity $E(T)$ and the frequency $\Omega(T)$ evolve on the slow-time scale $T\equiv\gamma t$, according to

$$E(T)\equiv\sqrt{\frac{1-\epsilon e^{-T}}{1+\epsilon e^{-T}}}, \tag{17a}$$

$$\Omega(T)=\sqrt{\mu^2(T)-\epsilon^2}=\sqrt{e^{2T}-\epsilon^2}. \tag{17b}$$

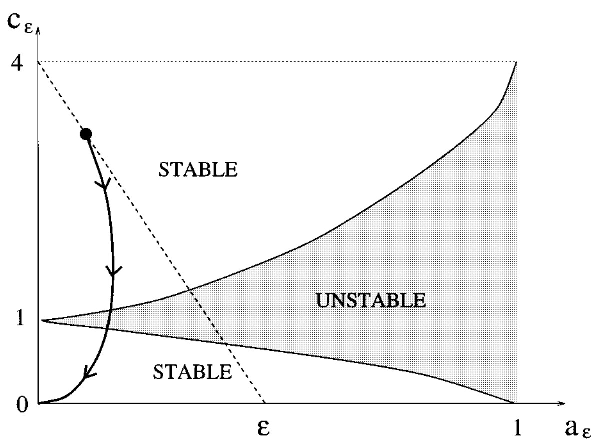

FIG. 1.

Inserting expressions (16a)–(16c) in (15), one obtains an explicit equation for the wave amplitude:

$$\left(\partial_t^2+\frac{c_\epsilon(T)+4a_\epsilon(T)\cos[(2/\gamma)\int_0^T\Omega(s)ds+2\xi_0]}{1+a_\epsilon(T)\cos[(2/\gamma)\int_0^T\Omega(s)ds+2\xi_0]}\times\Omega^2(T)\right)W-\gamma\,\partial_t W+O(\gamma\epsilon,\gamma^2)=0, \tag{18}$$

with

$$W=\widetilde{W}|k|^2, \tag{19a}$$

$$a_\epsilon(T)=\epsilon\,\frac{(k_0^\perp)^2e^{-T}}{(k_0^\perp)^2+e^{-3T}\sqrt{1-\epsilon^2e^{-2T}}}, \tag{19b}$$

$$c_\epsilon(T)=4\,\frac{e^{-3T}\sqrt{1-\epsilon^2e^{-2T}}}{(k_0^\perp)^2+e^{-3T}\sqrt{1-\epsilon^2e^{-2T}}}. \tag{19c}$$

We present in this section a naive approach to solve Eq. (18). Suppose we only consider the leading order in (18), i.e. the last term proportional to γ is disregarded. Equation (18) can then be transformed into a Hill equation with slowly evolving coefficients,

$$[1+a_\epsilon(T)\cos(2\xi)]\partial_\xi^2W+[c_\epsilon(T)+4a_\epsilon(T)\cos(2\xi)]W=0, \tag{20}$$

where the fast time variable,

$$\xi=\frac{1}{\gamma}\int_0^T\Omega(s)ds+\xi_0, \tag{21}$$

is employed. When γ is zero, a_ϵ and c_ϵ are constants. One then recovers the evolution of three-dimensional perturbations superimposed on the two-dimensional elliptical flow obtained by Waleffe.[8] Figure 1 reproduced from Waleffe[7] shows the behavior of such solutions for various values of the parameters a_ϵ and c_ϵ. In the shaded area, perturbations grow exponentially and they are neutral everywhere else.

In the present study, the coefficients a_ϵ and c_ϵ are not constant but slowly evolve according to expressions (19b) and (19c) on a continuous curve sketched in Fig. 1. Because of the *slow* variation of a_ϵ and c_ϵ, it is possible to use a quasistatic approximation to understand the behavior of the solutions of Eq. (20). Based on Waleffe's results, the elliptical instability is only active when a_ϵ and c_ϵ evolve inside the shaded area. If an initial perturbation is such that $k_0^\perp$ is sufficiently small, the point (a_ϵ,c_ϵ) always crosses the unstable region (see Fig. 1) during a finite period of time that scales as $O(\epsilon/\gamma)$. In that area, its amplitude W grows with an $O(\epsilon)$ rate. Therefore an exponential factor of the form $e^{K\epsilon^2/\gamma}$ can be expected for the global amplification. The basic flow is hence stable if $\epsilon^2\ll\gamma$ and unstable if $\gamma\ll\epsilon^2$. In the subsequent section, a more precise analysis of the solutions is performed and an expression for the global amplification factor is provided.

V. PERTURBATION THEORY

If one agrees with the above analysis, the effects of eccentricity and stretching are captured at the same order when γ scales with ϵ^2. In order to study the stability changes, it is therefore natural to consider the following scaling:

$$\gamma=\gamma_0\epsilon^2;\quad \gamma_0=O(1). \tag{22}$$

In the limit $\epsilon\ll1$, the coefficient a_ϵ given by (19b) remains small and the elliptical instability only occurs when c_ϵ is near 1 (see Fig. 1). This happens when T is in the neighborhood of the critical time T_c such that

$$c_\epsilon(T_c)=1. \tag{23}$$

Note that condition (23) express the fact that elliptical resonance occurs when the angle between $\mathbf{k}$ and the k_z axis is near $\pi/3$. A leading-order approximation for T_c is immediately obtained from (19c):

$$T_c=\frac{2}{3}\ln\left(\frac{\sqrt{3}}{k_0^\perp}\right). \tag{24}$$

Using the scaling (22), the expansion with respect to ϵ of Eq. (18) yields

$$[1+\epsilon a_0\cos(2\xi)]\partial_t^2W-\epsilon^2\gamma_0\,\partial_tW+[c_0+\epsilon^2c_2+4\epsilon a_0\cos(2\xi)]\Omega^2W+O(\epsilon^3)=0, \tag{25}$$

where Ω, W, ξ have been defined in (17b), (19a), (21), and

$$c_0=4\,\frac{e^{-3T}}{(k_0^\perp)^2+e^{-3T}}, \tag{26a}$$

$$a_0=\frac{4-c_0}{4\Omega}, \tag{26b}$$

$$c_2=-\frac{c_0(4-c_0)}{8\Omega^2}. \tag{26c}$$

If one searches for a local approximation of solutions of (25), it is necessary to discriminate between two possibilities: the nonresonant case $T\neq T_c$ and the resonant case $T=T_c$. Let us first consider the nonresonant case.

A. Nonresonant regions $T \neq T_c$

If T plays the role of a parameter, Eq. (25) is a linear ODE with periodic coefficients of period π/Ω. For such an equation, Floquet's theory shows that there exist solutions of the form $e^{\theta_l t} f_l(t)$, where the *characteristic exponents* θ_l are complex numbers and f_l periodic functions with the period of the coefficients. If the characteristic exponents are expanded in power series in ϵ, one obtains, at leading order, two independent solutions W_1, W_2 associated with two different values $\theta_1 \sim i\Omega\sqrt{c_0}$, $\theta_2 \sim -i\Omega\sqrt{c_0}$. Since Eq. (25) is invariant under complex conjugation, W_2 can actually be identified with W_1^*.

When the coefficients are slowly varying as in Eq. (25), the solutions may be expected to remain similar. As a consequence, a WKBJ uniform approximation is sought in the form

$$W_1 = \exp\left(\frac{i}{\gamma_0 \epsilon^2}\int_0^T \Omega(r)\sqrt{c_0(r)}\,dr\right) \times\left[A_{1_0} + \epsilon\left(A_{1_1} + B_1 \exp\left(-\frac{2i}{\gamma_0\epsilon^2}\int_0^T \Omega(r)\,dr\right) + C_1 \exp\left(\frac{2i}{\gamma_0\epsilon^2}\int_0^T \Omega(r)\,dr\right)\right) + O(\epsilon^2)\right]. \quad (27)$$

Once expression (27) is inserted in Eq. (25), the identification of oscillating terms at frequency $\Omega(T)(\sqrt{c_0(T)} \pm 2)$ yields, at order ϵ, the following relations:

$$B_1(T) = \frac{[4 - c_0(T)]^2}{32\Omega(T)[1 - \sqrt{c_0(T)}]}\, A_{1_0}(T), \quad (28a)$$

$$C_1(T) = \frac{[4 - c_0(T)]^2}{32\Omega(T)[1 + \sqrt{c_0(T)}]}\, A_{1_0}(T). \quad (28b)$$

At order ϵ^2, another equation for A_{1_0} is obtained by identifying terms oscillating at the frequency $\Omega(T)\sqrt{c_0(T)}$. Using expressions (28a) and (28b), it yields

$$A_{1_0}(T) = K_1(T) \times \exp\left(\frac{i}{\gamma_0}\int^T \frac{[4 - c_0(r)]^4}{4^4\Omega(r)[1 - c_0(r)]\sqrt{c_0(r)}}\,dr\right), \quad (29)$$

with

$$K_1(T) = \frac{e^{T/2}}{[\Omega\sqrt{c_0}]^{1/2}} \times \exp\left(-\frac{i}{\gamma_0}\int_0^T \frac{\sqrt{c_0(r)}[4 - c_0(r)]}{16\Omega(r)}\,dr\right). \quad (30)$$

The integral (29) is defined in an interval that belongs either to the set $T < T_c$ or to $T > T_c$, where $c_0(T) \neq 1$. Such expressions provide uniformly valid approximations for W_1 away from the resonance. As a consequence, any real solution of Eq. (25) is written, for all $T < T_c$, as

$$W = W^- \equiv A^- W_1 + \text{c.c.}, \quad \text{for } T < T_c, \quad (31)$$

where A^- is a complex constant. In the postresonant regime $T > T_c$, this solution takes the form

$$W = W^+ \equiv (A^+ W_1 + \text{c.c.}). \quad (32)$$

The complex constant A^+ is determined by matching expressions (31) and (32) through the resonant region at T_c.

B. Resonant region $T \approx T_c$

In the neighborhood of T_c, the characteristic local scale can be deduced from the analysis of (27) and (28a). Indeed, as soon as $|T - T_c| = O(\epsilon)$, the third term in (27) is of the same order or larger than the first term and both terms oscillate at the same frequency $\Omega_c \equiv \Omega(T_c)$. Since (28a) and (28b) and (29) were obtained assuming both order and frequency separations, these expressions are no longer justified as soon as $|T - T_c| = O(\epsilon)$: we therefore look for a different evolution on the local scale $\overline{T} \equiv (T - T_c)/\epsilon$.

As can be guessed from the behavior of W^- as $T \to T_c^-$ [see expression (A1a) given in the Appendix], the local approximation reads as

$$W = \overline{W} \equiv \overline{A}(\overline{T}) e^{i\Omega_c t} + \text{c.c.} + O(\epsilon), \quad (33)$$

where $\Omega_c \sim \sqrt[3]{3/(k_0^\perp)^2}$. The amplitude $\overline{A}$ satisfies

$$\frac{\partial \overline{A}}{\partial \overline{T}} = -\frac{i\Omega_c}{8\gamma_0}\,\overline{T}\,\overline{A} + \frac{9i}{16\gamma_0}\,\overline{A}^* e^{i\Omega_c \overline{T}^2/\gamma_0}. \quad (34)$$

This equation has been obtained from Eq. (25) by expanding all the coefficients of (25) in powers of ϵ:

$$c_0 = 1 - \tfrac{9}{4}\epsilon\overline{T} + O(\epsilon^2), \quad (35a)$$

$$\Omega = \Omega_c[1 + \epsilon\overline{T}] + O(\epsilon^2), \quad (35b)$$

$$\xi = \Omega_c[(t - t_c) + (\overline{T})^2/\gamma_0] + O(\epsilon), \quad (35c)$$

and identifying all $O(\epsilon)$ terms.

Equation (34) is transformed by the change of function,

$$\overline{A}(\overline{T}) = \widetilde{A}(\overline{T}) e^{-i\Omega_c \overline{T}^2/2\gamma_0}, \quad (36)$$

into a parabolic cylinder equation,

$$\frac{\partial^2 \widetilde{A}}{\partial \overline{T}^2} + \left[\left(\frac{9\Omega_c}{8\gamma_0}\right)^2 \overline{T}^2 - \left(\frac{9}{16\gamma_0}\right)^2 + \frac{9i\Omega_c}{8\gamma_0}\right]\widetilde{A} = 0, \quad (37)$$

the general solution of which can be expressed in terms of parabolic cylinder functions $\mathbf{D}_\alpha$ (see Abramowitz and Stegun[16]) as

$$\tilde{A}(\overline{T}) = \lambda \mathbf{D}_\alpha(\beta e^{5i\pi/4}\overline{T}) + \eta \mathbf{D}_{-\alpha-1}(\beta e^{3i\pi/4}\overline{T}), \quad (38)$$

with

$$\alpha \equiv \frac{9i}{64\gamma_0\Omega_c}, \quad (39a)$$

$$\beta \equiv \frac{3}{2}\sqrt{\frac{\Omega_c}{\gamma_0}}. \quad (39b)$$

Through the matching of the local approximation (33) with expressions (31) and (32), one obtains the constants λ, η, and A^+. The details of the matching procedure and the complete expressions are given in the Appendix.

Finally, the above computation yields the gain of amplitude across the region of resonance $G = |A^+/A^-|$:

$$G = e^{9\pi/(64\Omega_c\gamma_0)}\left|1 + f_\epsilon(\Omega_c, \gamma_0)\,\frac{(A^-)^*}{A^-}\right|, \quad (40)$$

where

$$|f_\epsilon(\Omega_c, \gamma_0)| = \sqrt{1 - e^{-9\pi/(32\Omega_c\gamma_0)}}. \quad (41)$$

VI. DISCUSSION

The above study generalizes the elliptical instability of two-dimensional vortices in the case where such vortices are subjected to an axial stretching. Its main result implies that the elliptical instability can be suppressed when the stretching γ reaches a high enough value compared to the magnitude ϵ of the asymmetry due to the average strain field. This is clearly seen on the inviscid amplitude gain G computed in Eq. (40). Indeed the parameter $1/(\gamma_0\Omega_c) = \epsilon^2/(\gamma\Omega_c)$ appears in G in an exponential factor. If $\gamma\Omega_c \ll \epsilon^2$, the perturbation amplitude can then be made as large as wanted on a time scale $t = O[1/(\gamma\Omega_c)]$. This inviscid instability process is nothing but the usual elliptical instability phenomenon generalized in the presence of small stretching. On the other hand, when the stretching becomes important compared to the destabilizing effect of eccentricity i.e. $\Omega_c\gamma \gg \epsilon^2$, there is no gain of amplitude since G tends to 1. As a result, the vortex structure is not affected by the elliptical instability. Actually perturbations are ''amplified'' during a period of time that is too short for them to reach a large amplitude.

As for the two-dimensional case,[8] the inviscid gain G does not depend on the initial amplitude of the wave vector. In the limit of small viscosity, small and large wave numbers are destabilized in the same manner. The effects of viscosity can be obtained by taking into account the exponential damping factor $\exp(-\nu\int_0^t |k(s)|^2\, ds)$ in expression (14) for the perturbations. The dependence of this factor on the wave number k implies that the maximum amplification gain $G_{\text{tot}}(k)$ is obtained for wave numbers $k \to 0$ and that a cutoff wave number k_{cut} is present for which perturbations of wave number $k \geq k_{\text{cut}}$ are damped by viscous effects. One then qualitatively recovers the result[9] that viscosity does not modify the evolution of large structures. The exact cutoff wave number for which viscous damping stabilizes any perturbation of wave number $k \geq k_{\text{cut}}$ is, however, more difficult to compute than for the 2-D case. This requires an evaluation of the maximum amplification gain $G_{\text{tot}}(k)$ for the set of perturbations associated with a given wave number k. Such perturbations are only amplified due to inviscid processes in the resonant region during a time interval $[T_1, T_2]$ for which the coefficient $(9\Omega_c/8\gamma_0)^2\overline{T}^2 - (9/16\gamma_0)^2$ in Eq. (37) is positive. The smallest viscous correction to the inviscid gain G can then be approximately evaluated by computing $\exp(-\nu\int^t |k(s)|^2\, ds)$ during the period of time $T_2 - T_1$. This yields

$$G_{\text{tot}}(k) \approx e^{9\pi\epsilon^2/(64\Omega_c\gamma)} e^{-\nu k^2\epsilon/(\Omega_c\gamma)},$$

and a cutoff wave number $k_{\text{cut}} \approx \sqrt{9\pi\epsilon/(64\nu)}$ for which $G_{\text{tot}} = 1$.

Our stability analysis is pertinent for any experimental vortex that is not viscous dominated. Indeed in that case, the core size L is much larger than the viscous scale $\sqrt{\nu/\gamma}$, which guarantees that $L \gg 1/k_{\text{cut}}$ when the inviscid instability condition $\epsilon^2 \gg \gamma\Omega_c$ is satisfied. A continuum of wave vectors corresponding to all the scales between $1/k_{\text{cut}}$ and L is then always destabilized in the instability process.

It is argued that the present study might explain recent experimental observations of vortex structures in turbulent flows. In real flow configurations, the vorticity field is subjected to fluctuating stretching and asymmetric strain. As described by our basic flow solution, stretching tends to concentrate the vorticity and generate localized vortical structures. If during this process, the vorticity fields is also subjected to a strong asymmetric strain, or if during a certain period of time the asymmetry is simply too large compared to the stretching, the structure in formation is destroyed by the elliptical instability. This could perhaps explain the rapid disappearance of vortex filaments observed by Cadot *et al.*[1] Finally, if the vortical structure is not destabilized, the vorticity concentration process is ultimately stopped by viscosity when the core size becomes $O(\sqrt{\nu/\gamma})$. The vortex is then expected to relax to a stationary solution controlled by viscosity (see, e.g., Moffatt *et al.*[5]). An extension of this work would consist in studying the stability of such a resulting vortex.

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APPENDIX: MATCHING

In this section, pre- and post-resonance amplitudes (31) and (32) are matched to the local approximation (33) valid in a $O(\epsilon)$ neighborhood of T_c.

The asymptotic behavior of W^- as $T \to T_c^-$ and W^+ as $T \to T_c^+$ can be obtained from (27)–(32). In terms of the local variable $\overline{T} = (T - T_c)/\epsilon$, it reads

$$W^- \underset{T\to T_c^-}{\sim} e^{i\Omega_c t} e^{i\Omega_c \overline{T}^2/(2\gamma_0)} \left(A^- K_1^c e^{iR_c} |\epsilon \overline{T}|^{9i/(64\Omega_c\gamma_0)} e^{-9i\Omega_c \overline{T}^2/(16\gamma_0)} + \frac{(A^-)^*[K_1^c]^*}{4\Omega_c \overline{T}} e^{-iS_c} |\epsilon \overline{T}|^{-9i/(64\Omega_c\gamma_0)} e^{9i\Omega_c \overline{T}^2/(16\gamma_0)} \right) + \text{c.c.}, \tag{A1a}$$

$$W^+ \underset{T\to T_c^+}{\sim} e^{i\Omega_c t} e^{i\Omega_c \overline{T}^2/(2\gamma_0)} \left(A^+ K_1^c e^{iR_c} |\epsilon \overline{T}|^{9i/(64\Omega_c\gamma_0)} e^{-9i\Omega_c \overline{T}^2/(16\gamma_0)} + \frac{(A^+)^*[K_1^c]^*}{4\Omega_c \overline{T}} e^{-iS_c} |\epsilon \overline{T}|^{-9i/(64\Omega_c\gamma_0)} e^{9i\Omega_c \overline{T}^2/(16\gamma_0)} \right] + \text{c.c.}, \tag{A1b}$$

where

$$R_c \equiv \frac{1}{\epsilon^2\gamma_0} \int_0^{T_c} \Omega(r)\sqrt{c_0(r)}\,dr - \frac{\Omega_c T_c}{\epsilon^2 \gamma_0}, \tag{A2a}$$

$$S_c \equiv \frac{1}{\epsilon^2\gamma_0} \int_0^{T_c} \Omega(r)[\sqrt{c_0(r)} - 2]\,dr + \frac{\Omega_c T_c}{\epsilon^2 \gamma_0}, \tag{A2b}$$

$$K_1^c \equiv K_1(T_c). \tag{A2c}$$

In order to find the constants λ and η in (38), we need to compute the asymptotic behavior of the local approximation (36) as $\overline{T}\to-\infty$ and $\overline{T}\to+\infty$. One can deduce this from the expansions of the parabolic cylinder function $\mathbf{D}_\alpha(z)$ as $z\to\infty$ (see pp. 131–132 in Ref. 15):

$$\tilde{A}(\overline{T}) \underset{\overline{T}\to-\infty}{\sim} e^{i\pi\alpha/4}(\lambda|\beta\overline{T}|^{\alpha} e^{i(\beta\overline{T})^2/4} + e^{i\pi/4}\eta|\beta\overline{T}|^{-\alpha-1} e^{i(\beta\overline{T})^2/4}), \tag{A3a}$$

$$\tilde{A}(\overline{T}) \underset{\overline{T}\to+\infty}{\sim} \left(\eta e^{-i\pi\alpha/4} \frac{\sqrt{2\pi}}{\Gamma(\alpha+1)} + \lambda e^{-3i\pi\alpha/4} \right) |\beta\overline{T}|^{\alpha} e^{i(\beta\overline{T})^2/4}, + \left(\lambda e^{-i\pi(\alpha+1)/4} \frac{\sqrt{2\pi}}{\Gamma(-\alpha)} + \eta e^{-3i\pi(\alpha+1)/4} \right) |\beta\overline{T}|^{-\alpha-1} e^{i(\beta\overline{T})^2/4}. \tag{A3b}$$

Matching the outer solutions (A1a) and (A1b) with the inner solution (33), (36) and (A3a) and (A3b), we finally obtain

$$\lambda = \left|\frac{\epsilon}{\beta}\right|^{i\alpha_i} K_1^c e^{iR_c} e^{-\pi\alpha_i/4} A^-, \tag{A4a}$$

$$\eta = e^{-i\pi/4}\sqrt{\alpha_i}\left|\frac{\epsilon}{\beta}\right|^{-i\alpha_i} [K_1^c]^* e^{-iS_c} e^{-\pi\alpha_i/4}[A^-]^*, \tag{A4b}$$

$$\eta e^{\pi\alpha_i/4} \frac{\sqrt{2\pi}}{\mathbf{\Gamma}(i\alpha_i+1)} + \lambda e^{3\pi\alpha_i/4} = \left|\frac{\epsilon}{\beta}\right|^{i\alpha_i} K_1^c e^{iR_c} e^{-\pi\alpha_i/4} A^+, \tag{A4c}$$

$$\lambda e^{\pi\alpha_i/4} \frac{\sqrt{2\pi}}{\mathbf{\Gamma}(-i\alpha_i)} - \eta i e^{3\pi\alpha_i/4} = e^{-i\pi/4}\sqrt{\alpha_i}\left|\frac{\epsilon}{\beta}\right|^{-i\alpha_i} [K_1^c]^* e^{-iS_c}[A^+]^*. \tag{A4d}$$

The above expressions are compatible and yield

$$A^+ = e^{9\pi/(64\Omega_c\gamma_0)}[A^- + f_\epsilon(\Omega_c,\gamma_0)(A^-)^*], \tag{A5}$$

with

$$f_\epsilon(\Omega_c,\gamma_0) = e^{-i(R_c+S_c+\pi/4)} \frac{[K_1^c]^*}{K_1^c} \left[\frac{4\epsilon^2\gamma_0}{9\Omega_c}\right]^{-9i/(64\Omega_c\gamma_0)} \times \frac{3\sqrt{2\pi}e^{-9\pi/(128\Omega_c\gamma_0)}}{8\sqrt{\Omega_c\gamma_0}\mathbf{\Gamma}(1+9i/(64\Omega_c\gamma_0))}. \tag{A6}$$

Note that the modulus $|f_\epsilon(\Omega_c,\gamma_0)|$ simply reads as

$$|f_\epsilon(\Omega_c,\gamma_0)| = \sqrt{1-e^{-9\pi/(32\Omega_c\gamma_0)}}, \tag{A7}$$

since (see formula 6.1.31 in Ref. 16) $|\mathbf{\Gamma}(1+iy)|^2 = \pi y/\sinh \pi y$.

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