Short-wavelength instability of a vortex in a multipolar strain field

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The extension of the elliptical instability to a rotational flow with an n-fold symmetry is considered in this Brief Communication. Based on the geometrical optics approach, the short-wavelength stability analysis of the multipolar strained vortex described by the streamfunction (in polar coordinates) $\Psi_n = r^2/2-(p/n)r^n \cos(n\theta)$ is carried out for $n=3,4,5$ and $p\geqslant 0$. Both the growth rate and the wavevector properties of the most unstable wave are computed as a function of the real parameter $\varepsilon_n = p(2nC/(n-2))^{(n-2)/2}$ which characterizes the geometry of the streamline $\Psi_n = C$. For almost circular flows, i.e., in the small ε_n limit, exact estimates are also obtained by perturbation methods. © *1999 American Institute of Physics.* [S1070-6631(99)00602-9]

Due to the presence of boundaries or other vortices, two-dimensional nonviscous vortices are generically elliptical near their center. This characteristic makes them unstable by the so-called elliptical instability.[1,2] By contrast, if the external field exhibits particular symmetries, the vortex is not necessarily elliptical in its core and it may have an n-fold symmetry with $n>2$. In these cases, the elliptical instability is not present. It is then natural to address the stability of such vortices, and, in particular, to determine whether a destabilizing mechanism similar to the elliptical instability exists.

The elliptical instability could explain the three-dimensional (3D) transition of several flows,[3] such as wakes,[4] shear layers,[5] or other vortex flows.[6] Bayly[2] and Waleffe[7] gave a description of the instability mechanism in the context of a pure elliptical flow. Lifschitz and Hameiri[8] proposed an interesting Lagrangian approach, based on a geometrical optics method, to extend their stability analysis to more general configurations but restricted to small wavelengths. Their main idea was to construct 3D perturbations which are sufficiently localized such that their evolution is governed by an ordinary differential equation along the streamline. Both elliptical and hyperbolic instabilities[2,9] have been recovered by this approach.[8,10] It has also permitted new achievements for other nonuniform and time-varying flows (see Bayly *et al.*[11] and references therein).

In this Brief Communication, the geometrical optics theory[8] is used to analyze the stability properties of the 2D basic flow described by the streamfunction (in polar coordinates)

$$\Psi_n(r,\theta;p) = \frac{r^2}{2} - p\frac{r^n}{n}\cos(n\theta), \tag{1}$$

where n is an integer larger than 1 and p a real positive parameter. This flow is the superposition of a rotational field of uniform vorticity (first term) and of a multipolar potential strain field (second term) characterized by its order n and its strength p. It describes the core of a 2D nonviscous vortex in equilibrium with an external strain field which exhibits an n-fold symmetry. For $n=2$ and $n=3$, (1) is generic near the vortex center as it corresponds to the first terms in the Taylor expansion of the streamfunction with respect to the distance to the vortex axis. This is also the case for larger n if one assumes that the vorticity is sufficiently uniform in the vortex core. In particular, for all n, (1) is the streamfunction of the core of a stationary vortex patch (region of uniform vorticity) in an n-fold symmetrical strain field. In the following, we focus on local perturbations which grow within the basic flow described by (1). The external strain field generated by boundaries or distant vortices is not considered in the analysis.

The shape of the streamline $\Psi_n(r,\theta;p)=C$ is characterized by a single parameter,

$$\varepsilon_n \equiv p\left(\frac{2nC}{n-2}\right)^{(n-2)/2}, \tag{2}$$

which measures its asymmetry. In Fig. 1, the streamlines are displayed for four values of ε_n in the interval [0,1] and $n=3,4,5$. For $n=2$, the basic flow is the uniform elliptical flow studied by Bayly, Waleffe and others: all the streamlines are ellipses having the same eccentricity $\varepsilon_2 = p$. For $n\geqslant 3$, the basic flow is not uniform as ε_n varies with the label C of the streamline. For small ε_n, the streamline $\Psi_n = C$ is almost circular. It becomes more and more angular as ε_n increases up to $\varepsilon_n=1$ for which the streamline exhibits n singular points (corners). These singular points are, in fact, hyperbolic stationary points that play an important role in the stability properties, as we shall see below. For $\varepsilon_n>1$, the streamlines are no longer closed, so the analysis will be restricted to the range $0\leqslant\varepsilon_n\leqslant 1$.

In the geometrical optics stability theory,[8,11] perturbations are sought in the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}(\mathbf{x},t)\exp\left(\frac{i}{\varepsilon}\Phi(\mathbf{x},t)\right), \tag{3}$$

where the characteristic wavelength ε is the small parameter used for the asymptotic analysis. Substituting expression (3)

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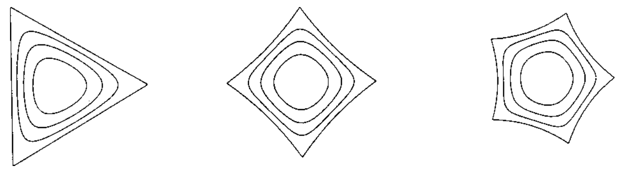

FIG. 1. Streamlines for $n=3,4,5$ (from left to right). The outer streamline is obtained for $\varepsilon_n=1$.

in the linearized Euler equations and equating terms of the same order lead to a set of equations which is written in Lagrangian form as follows:[8]

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}=\mathbf{U}_n(\mathbf{X},t), \tag{4a}$$

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t}=-(\nabla\mathbf{U}_n)^T(\mathbf{X},t)\mathbf{k}, \tag{4b}$$

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}=\left(\frac{2\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}-\mathcal{I}\right)\nabla\mathbf{U}_n(\mathbf{X},t)\mathbf{a}, \tag{4c}$$

where $\mathcal{I}$ is the identity matrix, $\mathbf{U}_n(\mathbf{x},t)$ is the velocity field associated with (1) and $\mathbf{k}=\nabla\Phi$ is the (local) wavevector along the Lagrangian trajectory $\mathbf{x}=\mathbf{X}(t)$. In addition, $(\mathbf{X},\mathbf{k},\mathbf{a})$ satisfies an initial condition $(\mathbf{X}_0,\mathbf{k}_0,\mathbf{a}_0)$ compatible with the incompressibility condition $\mathbf{k}_0.\mathbf{a}_0=0$. The main point of the theory[8] is that the existence of an unbounded amplitude $\mathbf{a}(t)$ provides a sufficient condition of instability for nonviscous flows. Viscous effects can easily be added[5,8] and the same result holds as long as the characteristic wavelength ε is larger than $\sqrt{\nu/s}$ where ν is the kinematic viscosity and s the maximum growth rate of $\mathbf{a}$. Here, ν is assumed sufficiently small such that the nonviscous system (4a–c) can be used for any bounded $\mathbf{k}$. Our purpose is to analyze the behavior of the amplitude $\mathbf{a}$ along the streamline $\Psi_n=C$. More precisely, we are going to compute the maximum mean growth rate of $\mathbf{a}$ during a turnover period along the streamline.

First, the streamline equation is solved numerically to obtain the trajectory as $r=\sqrt{2nC/(n-2)}r_n(\theta;\varepsilon_n)$. This result is then used to reduce Eq. (4a) to

$$\frac{d\theta}{dt}=\frac{1}{r}\frac{\partial\Psi_n}{\partial r}(r,\theta)=1-\varepsilon_n r_n^{n-2}(\theta;\varepsilon_n)\cos(n\theta), \tag{5}$$

and the velocity gradient tensor $\nabla\mathbf{U}_n$ in (4b) and (4c) to

$$\nabla\mathbf{U}_n(\theta;\varepsilon_n)=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}+(n-1)\varepsilon_n r_n^{n-2}(\theta;\varepsilon_n)\times\begin{pmatrix}\sin[(n-2)\theta]&\cos[(n-2)\theta]&0\\ \cos[(n-2)\theta]&-\sin[(n-2)\theta]&0\\0&0&0\end{pmatrix}. \tag{6}$$

Equation (5) can be solved without loss of generality with the initial condition $\theta(0)=0$ but this choice governs the initial condition for the wavevector $\mathbf{k}$. Indeed, if $\mathbf{k}(0)\cdot\mathbf{U}_n(\mathbf{X}(0),0)\neq 0$, it is possible to show[12] that the wavevector projection onto the streamline is stretched by a constant factor at each revolution: the wavevector is then unbounded which gives a perturbation amplitude damped by viscosity for large time. For this reason, it is important to only consider initial conditions that satisfy $\mathbf{k}(0)\cdot\mathbf{U}_n(\mathbf{X}(0),0)=0$, i.e., $\mathbf{k}(0)\cdot\mathbf{e}_y=0$ in order to obtain periodic wavevectors. Moreover, Eq. (4c) being independent of $|\mathbf{k}|^2$, one may choose $\mathbf{k}(0)=(\sin\chi,0,\cos\chi)$ where χ is the initial angle of the wavevector with respect to the z axis (vorticity axis). Substituting the wavevector solution in (4c) leads to a linear equation with periodic coefficients which can be analyzed by Floquet theory. This classical theory has already been used by Bayly[2] in the elliptical case. It only requires the integration of (4c) in the interval $[0,T_n]$, where $T_n=T_n(\varepsilon_n)$ is the turnover time along the streamline, with the identity matrix as an initial condition. The maximum mean growth rate of the amplitude $\mathbf{a}$ is then given by $s(\varepsilon_n,\chi)=\max_l[\ln(|\mu_l|)/T_n]$ where μ_l, $l=1,2,3$ are the Floquet exponents, i.e., the eigenvalues of the final matrix at $t=T_n$. Moreover, as soon as $s>0$, the eigenmode associated with the maximum growth rate remains orthogonal to the wavevector for all t: the incompressibility condition is therefore automatically fulfilled for all exponentially growing perturbations.[2]

The integration is carried out for $0\leqslant\chi\leqslant\pi/2$, $0\leqslant\varepsilon_n\leqslant 1$ and $n=3,4,5$. Figures 2(a)–(c) show the (mean) growth rate level curve of the amplitude $\mathbf{a}$ in the (ε_n,χ) plane for each value of n. On these figures is also represented the wavevector angle $\chi_{\max}$ at which the maximum growth rate is

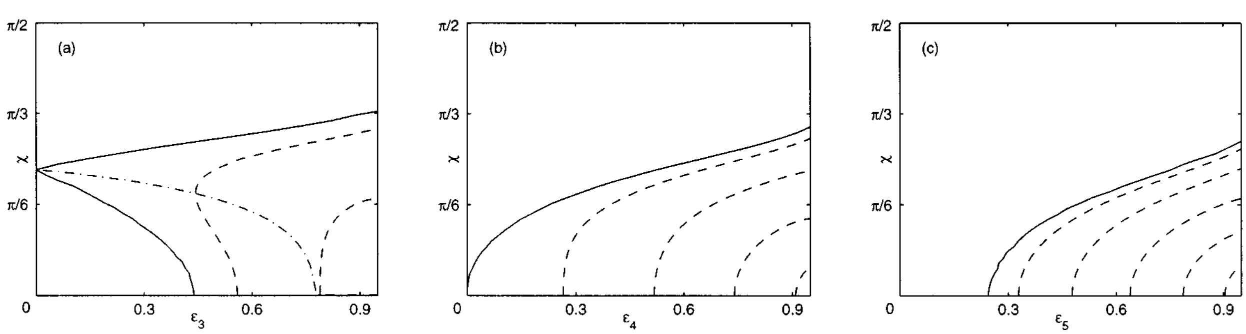

FIG. 2. The growth rate level curve in the (ε_n,χ) plane for $n=3$ (a), $n=4$ (b) and $n=5$ (c). The solid curve is the critical curve $(s=0)$. The step between two level curves (dashed curves) is 0.4. The dash–dotted curve in (a) gives the angle $\chi_{\max}$ at which the growth is maximized. In figures (b) and (c), $\chi_{\max}=0$.

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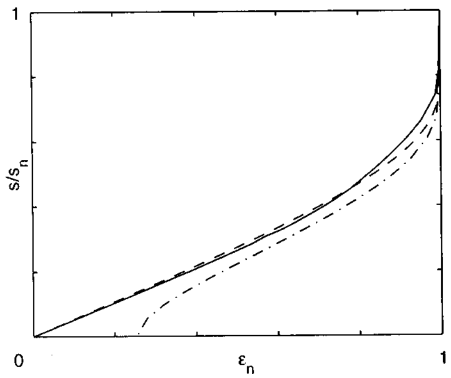

FIG. 3. Normalized maximum growth rate s/s_n versus ε_n for $n=3$ (solid line), $n=4$ (dashed line) and $n=5$ (dot–dashed line).

attained for a fixed ε_n. In Fig. 3 is displayed the normalized maximum growth rate s/s_n versus ε_n for each value of n. The largest value s_n of the growth rate is reached for $\varepsilon_n=1$. For this particular value of ε_n, the turnover time T_n becomes infinite and the Lagrangian trajectory stops at the first hyperbolic stagnation point it encounters on the streamline. For large time, the stability properties for $\varepsilon_n=1$ are thus given by those of the hyperbolic stagnation points on the streamline. Using the results[8,9,13] for hyperbolic stagnation points, the maximum growth rate is then obtained as $s_n=\sqrt{\partial_x^2\Psi_n\partial_y^2\Psi_n-(\partial_x\partial_y\Psi_n)^2}=\sqrt{n(n-2)}$, where the partial derivatives are calculated at one of the stagnation points.

Expressions for small ε_n are obtained by perturbation methods. For $\varepsilon_n=0$, the trajectory is circular and the Lagrangian evolution is the one of a solid-body rotation with an angular velocity $\mathbf{\Omega}=\mathbf{e}_z$. The short-wavelength perturbations (3) are in this case (local) *inertial* waves. Their characteristics are well known:[7] their wavevector rotates periodically with respect to the z-axis with the same frequency as the basic flow and with a constant inclination angle χ; their amplitude is also periodic with a frequency $\omega_0=2\mathbf{k}\cdot\mathbf{\Omega}/|\mathbf{k}|=2\cos\chi$. The instability mechanism for small ε_n can be understood as a simple phenomenon of resonance of the inertial waves with the multipolar strain field.[7,14] Indeed, at the next order in ε_n, the interaction of the inertial wave with the strain field generates waves of frequency $|n\pm\omega_0|$ which resonate with the inertial wave if their frequency equals ω_0. This yields a condition of resonance which is written as

$$n=4\cos\chi_n. \tag{7}$$

This equation selects resonant angles only if $n=2,3,4$. The value for $n=2$ has been given in Bayly.[2] For $n=3$ and $n=4$, the resonant angles are in perfect agreement, for small ε_n, with the computational results shown in Figs. 2(a),(b). For $n\geqslant 5$, there is no possible resonance at the first order. This is also in agreement with Figs. 2(c) and 3 which show that the basic flow for $n=5$ is stable for $\varepsilon_5\leqslant\varepsilon_{5c}\approx 0.25$.

The leading order growth rate of the instability can be calculated using a multiple scale analysis. After a long but straightforward calculation, we obtain a growth rate of the form $s=\sigma_n\varepsilon_n+O(\varepsilon_n^2)$ where

$$\sigma_2=9/16,$$
$$\sigma_3=49/(32\sqrt{3})\approx 0.884, \tag{8}$$
$$\sigma_4=3/2.$$

For $n=2$, Waleffe's result[7] for a weakly elliptical flow is recovered. For $n=3$ and $n=4$, this first order expression provides a very good estimate of the growth rate for $\varepsilon_n\leqslant 0.5$.

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