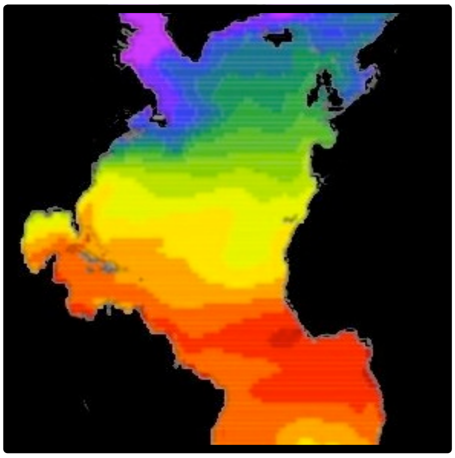
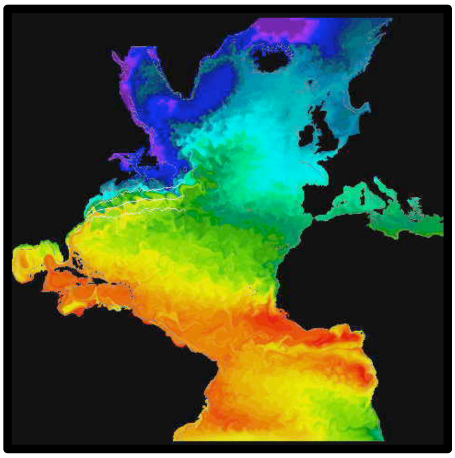
Geostrophic ocean eddies: impacts on the global circulation and a new framework for their parameterisation in ocean models

David Marshall, James Maddison *(University of Oxford)*
collaborators: Talia Tamarin, Pavel Berloff, Laure Zanna

1^0 (climate) resolution



$1/12^0$ resolution



(MICOM, University of Miami)

Structure

1. Very brief overview of:

- baroclinic instability;
- why geostrophic eddies matter for the global ocean and climate;
- Gent and McWilliams eddy parameterisation;
- alternative paradigm: isopycnal mixing of potential vorticity ... and caveats!

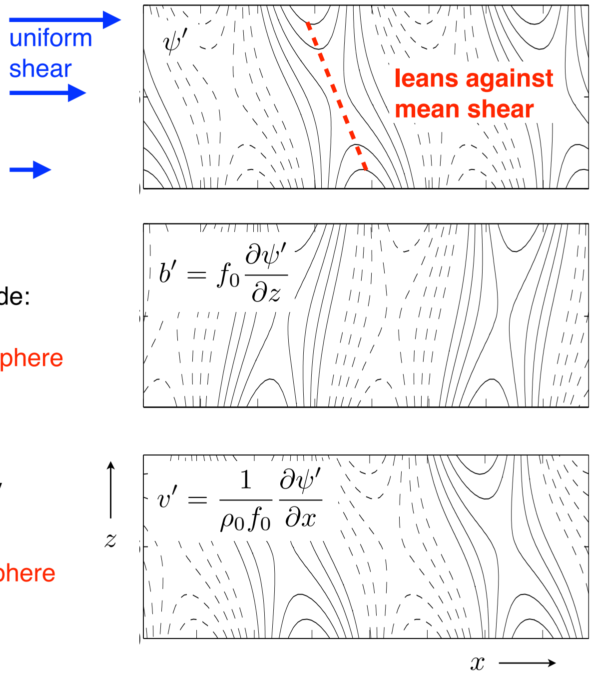
2. A new framework for parameterising ocean eddies:

- eddy stress tensor;
- geometric interpretation;
- Eady problem;
- ray tracing;
- potential vorticity mixing;
- future work and conclusions.

Eady (1949) model of baroclinic instability

- f-plane (neglect β effect)
- uniform stratification
- uniform shear
- opposite potential vorticity gradients at upper and lower boundaries

most **unstable** mode:



(figure: adapted from Vallis, 2006)

energy growth rate for most unstable mode:

$$0.61 \frac{f_0}{N_0} \frac{\partial u}{\partial z} \sim$$

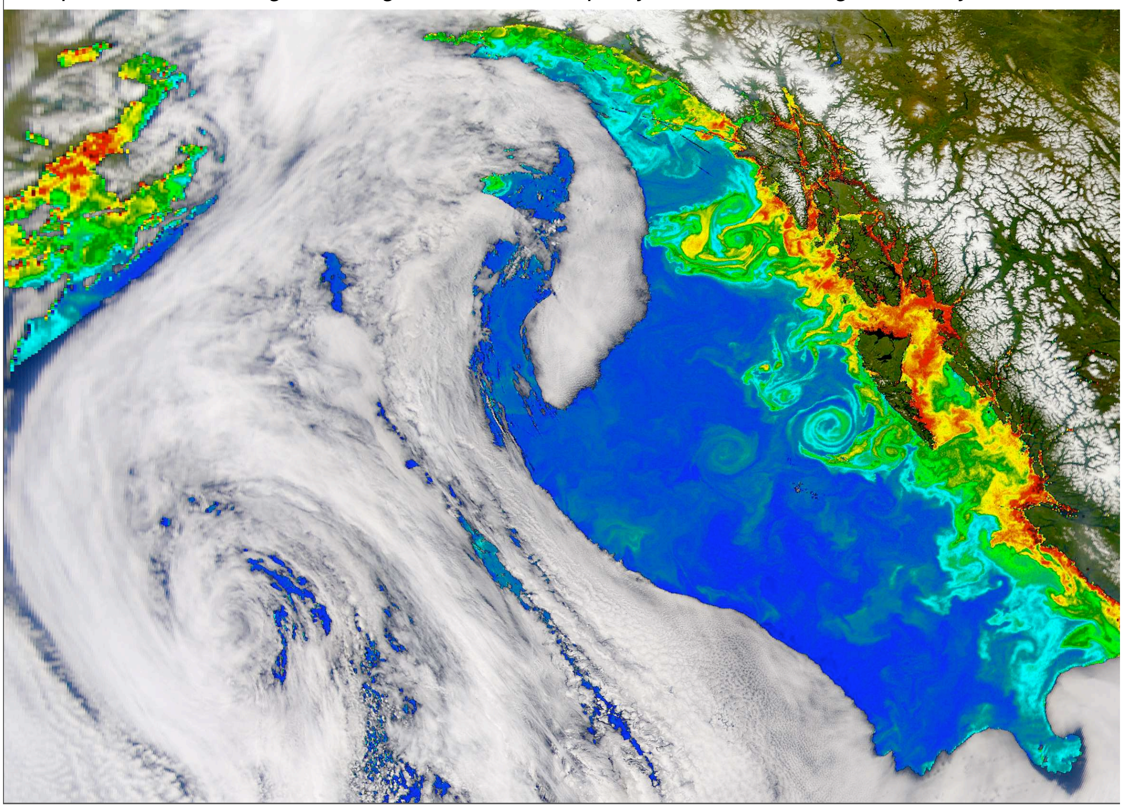
0.3 day^{-1} - atmosphere
0.03 day^{-1} - ocean

length scale of instability characterised by
Rossby deformation radius:

$$L_d = \frac{N_0 H}{f_0} \sim$$

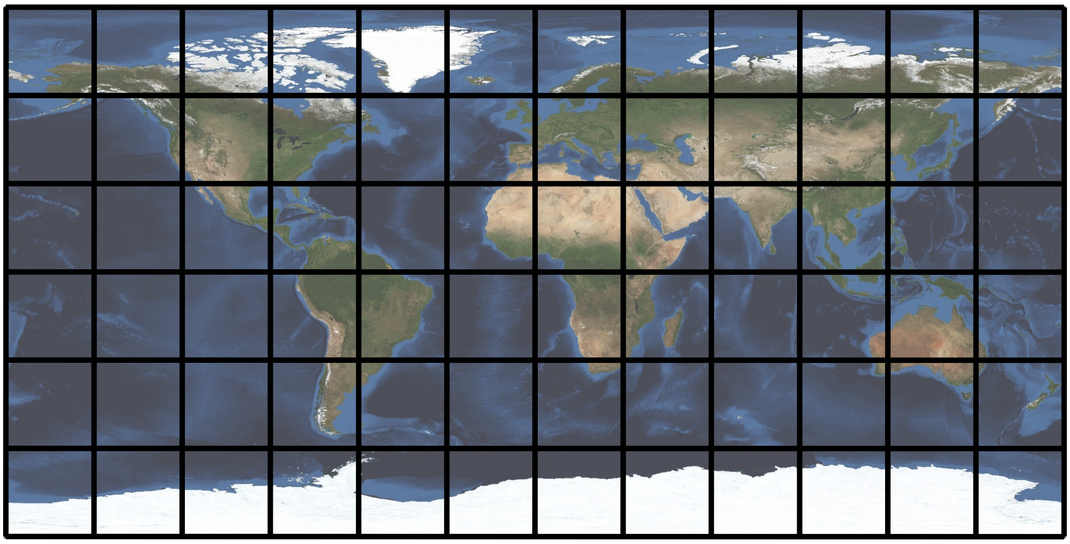
1000 km - atmosphere
50 km - ocean

Composite satellite image showing cloud cover and proxy for surface biological activity



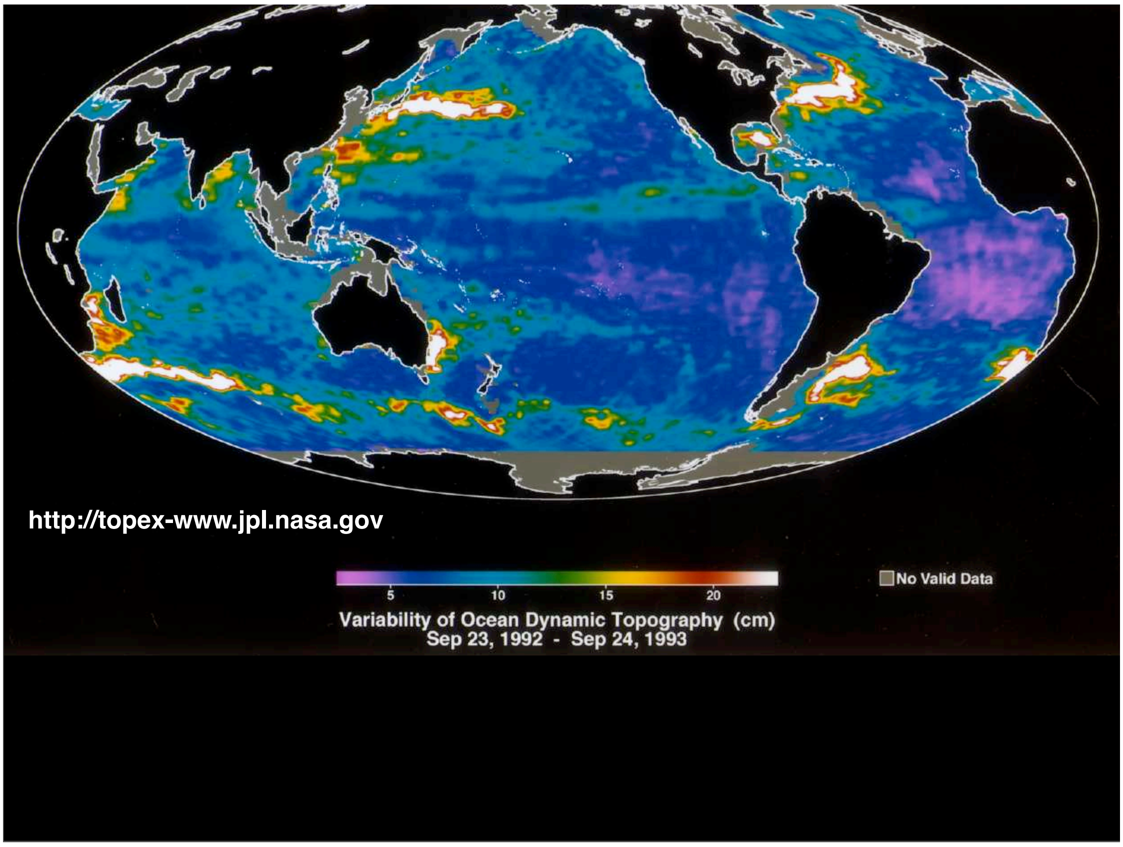
Challenge of resolving eddies in numerical ocean models

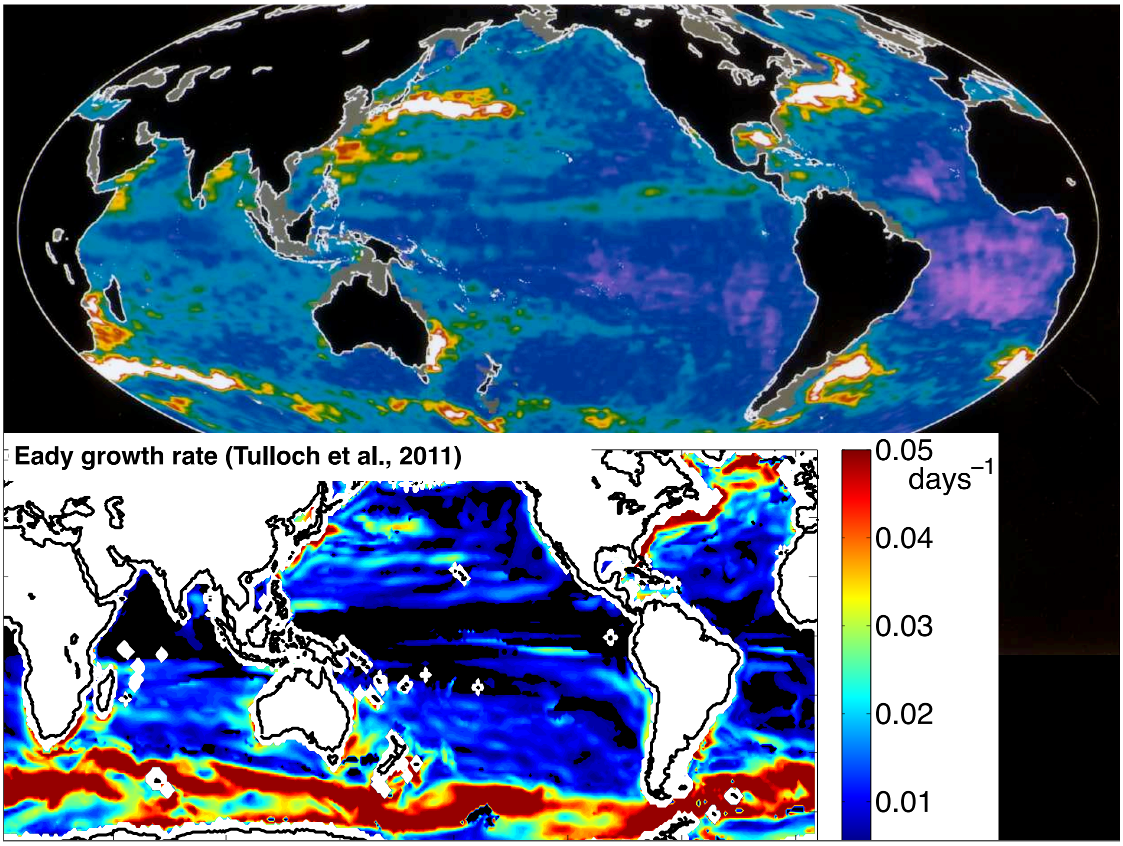
In terms of mesoscale eddy resolution, a 1° ocean model ~ 30° atmosphere model:



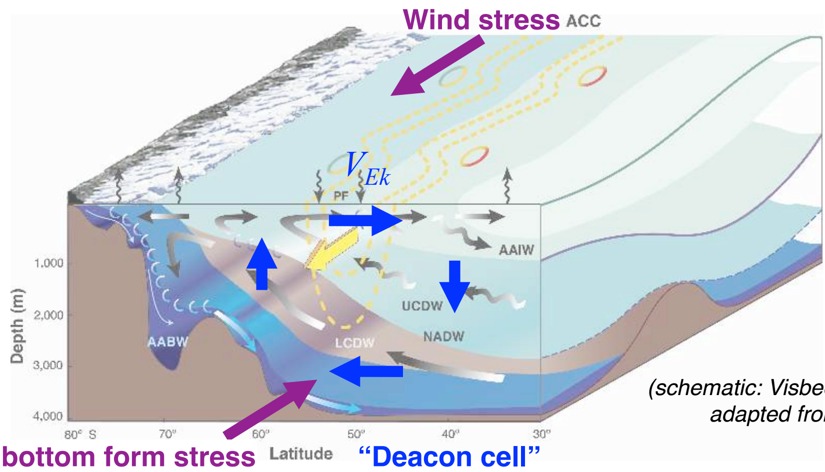
Conversely, a 1° atmosphere model ~ 1/30° ocean model:

(after Peter Killworth)



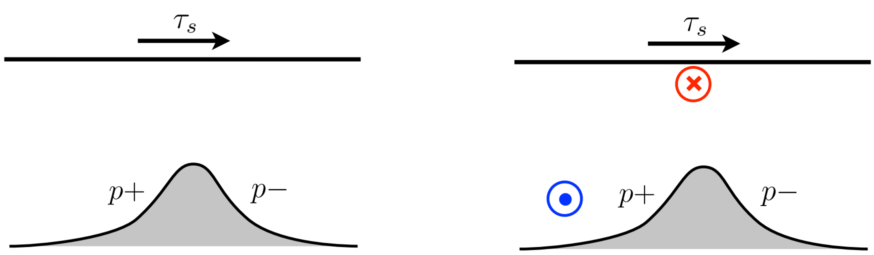


Impact of eddies: Antarctic Circumpolar Current



(schematic: Visbeck and Olbers 2004; adapted from Rintoul et al. 2001)

- Wind stress balanced by bottom form stress ($p_b \nabla H$) (Munk and Palmén, 1951)



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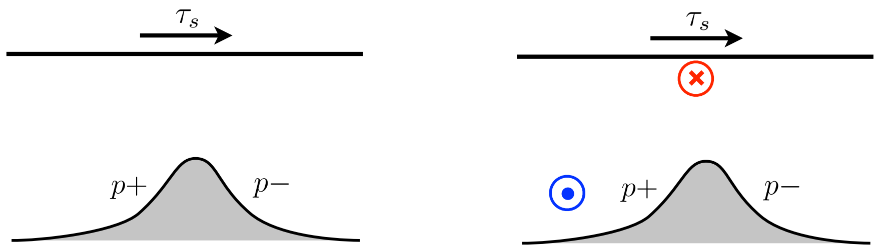
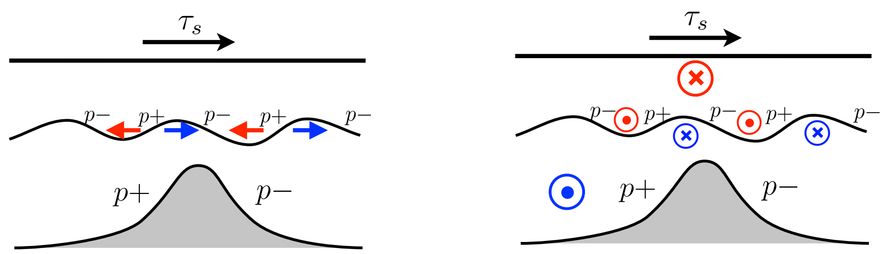


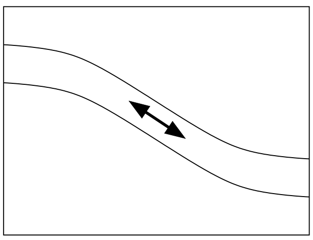
- Downward momentum transfer by eddy form stress (Rhines and Holland, 1979; Johnson and Bryden, 1989)

"non-acceleration conditions"

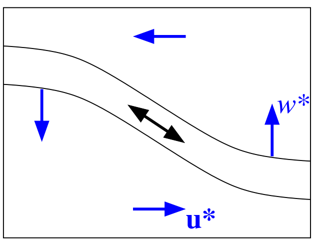


Gent and McWilliams (1990), Gent et al. (1995) adiabatic eddy closure

eddies diffuse tracers along isopycnals (Redi 1982)



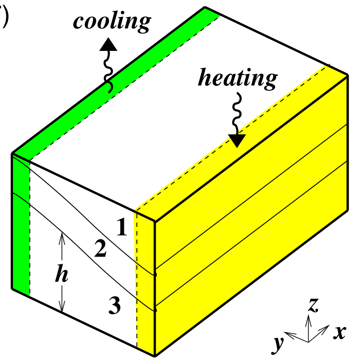
and advect by an *eddy transport velocity* (bolus velocity) - acts to flatten isopycnals

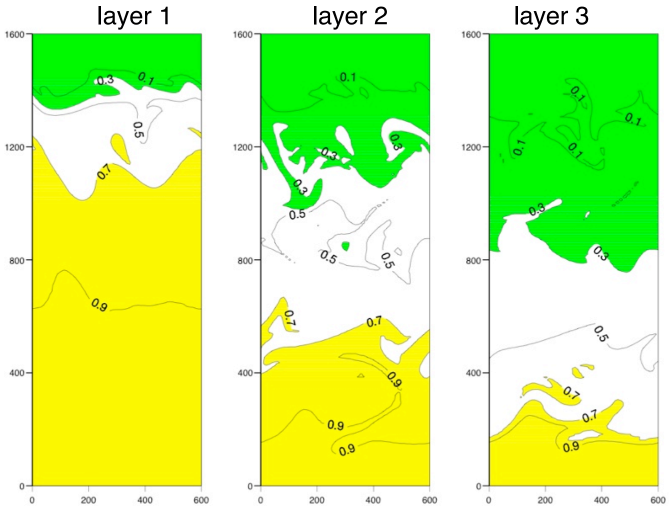


$$\mathbf{u}* = \frac{\partial}{\partial z}\left(\kappa \frac{\nabla b}{\partial b/\partial z}\right), \quad w* = -\nabla \cdot \left(\kappa \frac{\nabla b}{\partial b/\partial z}\right),$$

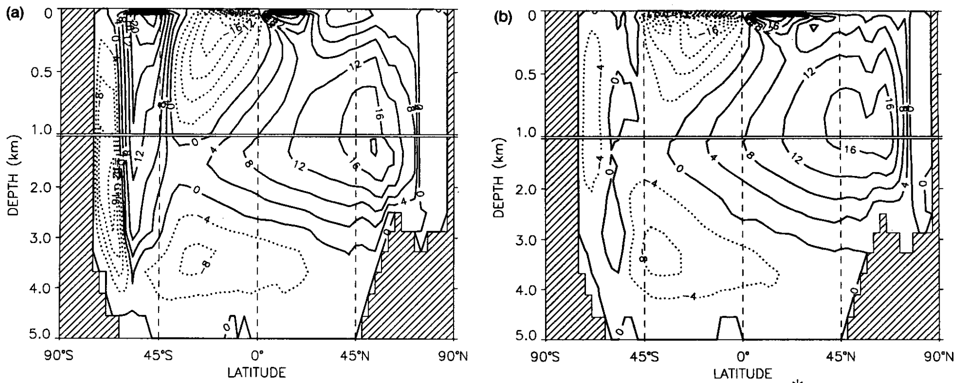
available potential energy sink -
parameterisation of baroclinic instability

Simple illustration in buoyancy-forced channel (Lee et al. 1997)





Meridional overturning circulation in a coarse-resolution ocean model:



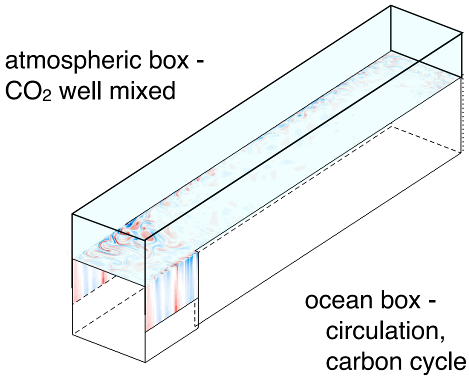
$\overline{\mathbf{u}}$ $\qquad$ $\overline{\mathbf{u}} + \mathbf{u}^*$

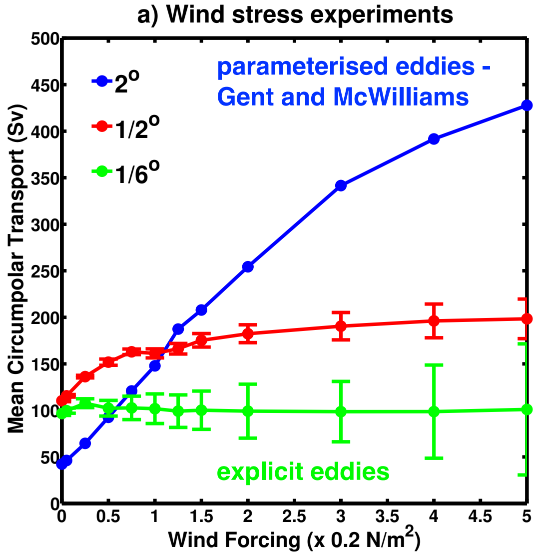
(Danabasoglu et al., 1994)

Many other improvements over previous non-adiabatic eddy closures:
- sharper thermocline;
- convection confined to places it is known to occur;
- removal of spurious upwelling in Gulf Stream;
- improved poleward heat transport.

Several extensions of Gent and McWilliams, mostly relating eddy diffusivity to mean fields, e.g., using Eady growth rate (Visbeck et al., 1997)

Note: explicit (partially-resolved) eddies behave very differently to Gent and McWilliams!





(Munday et al., 2013, in press, JPO)

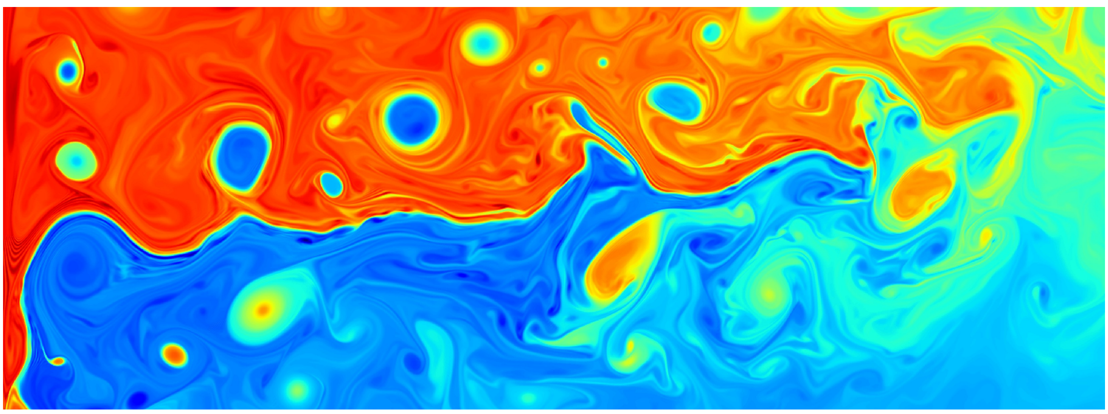
Alternative paradigm: potential vorticity mixing

often advocated ... rarely implemented!

Idea: potential vorticity $q = \dfrac{f + \xi}{h}$ is materially conserved in absence of forcing/dissipation:

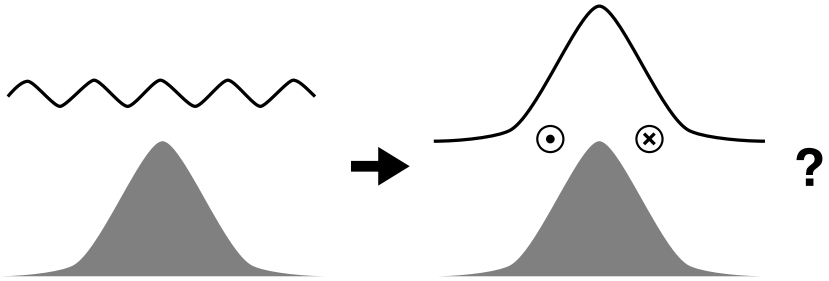
$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0$$

⇒ stirred and mixed along density surfaces?



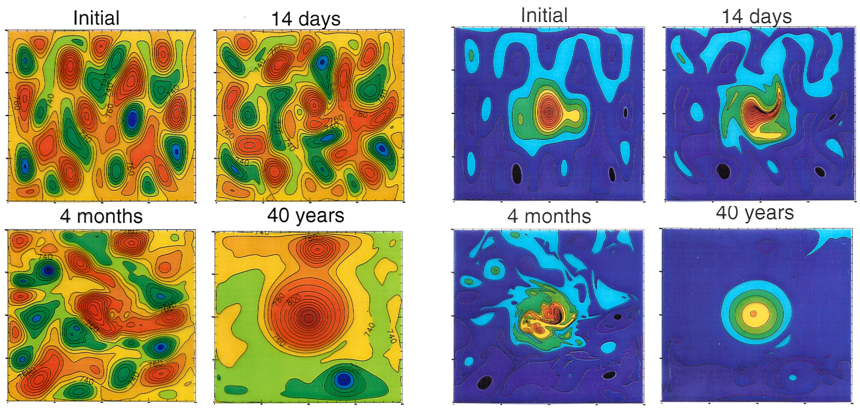
PV mixing problem 1: conservation of energy

e.g. , freely-decaying turbulence over a seamount (Adcock and Marshall, 2000)



geostrophic streamfunction

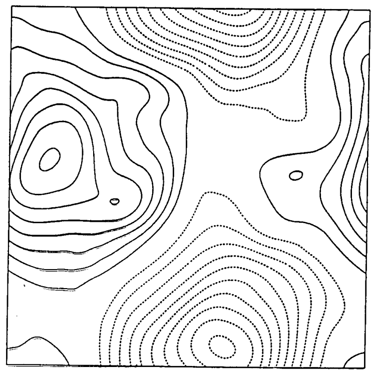
potential vorticity



Energy ~ conserved in geostrophic turbulence due to inverse cascade:

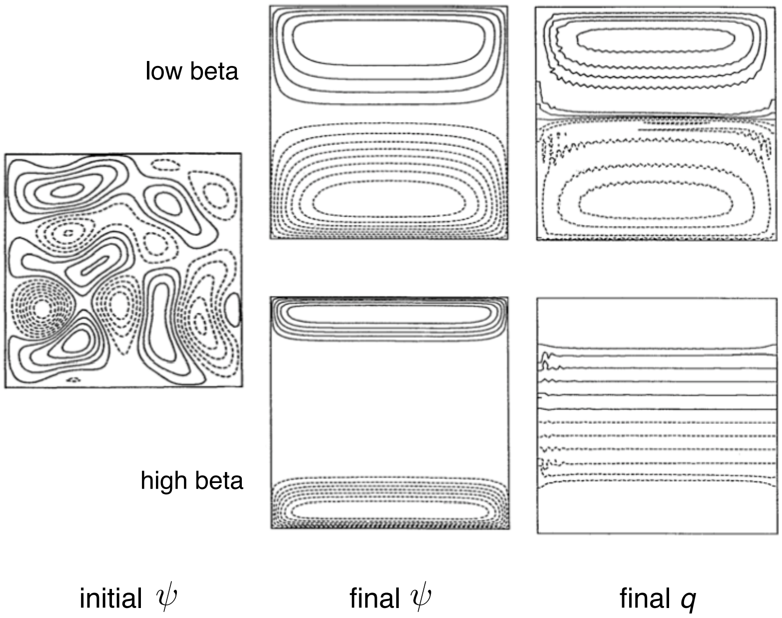
vorticity

streamfunction



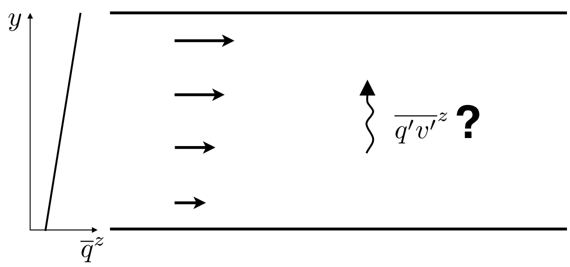
(calculation: Vallis and Maltrud)

Freely-decaying barotropic turbulence (Wang and Vallis, 1994)



PV mixing problem 2: conservation of momentum

e.g. , consider a quasigeostrophic periodic channel:



$$\iiint \overline{q'v'}\, dx\, dy\, dz = 0$$

conservation of momentum

not satisfied by down-gradient potential vorticity closure without constraints on eddy diffusivity (Green, 1970; J. Marshall, 1981)

suppose $dq/dy > 0$

⇒ down-gradient eddy closure, $\overline{q'v'} = -\kappa\, \partial q/\partial y$, only consistent if $\kappa = 0$ (i.e., no eddies!)

note: this is the Charney-Stern stability condition

Eddies mix potential vorticity along density surfaces ...
... subject to constraints of energy and momentum conservation

Goal:

Develop framework for **interpreting** and **parameterising** eddy potential vorticity fluxes in which the relevant **symmetries and conservation laws are preserved**.

Quasi-geostrophic equations

momentum: $$\frac{\partial \mathbf{u}_g}{\partial t} + \mathbf{u}_g \cdot \nabla \mathbf{u}_g + f_0 \mathbf{k} \times \mathbf{u}_{ag} + \beta y \, \mathbf{k} \times \mathbf{u}_g + \frac{\nabla p_{ag}}{\rho_0} = 0$$

buoyancy: $$\frac{\partial b}{\partial t} + \mathbf{u}_g \cdot \nabla b + w_{ag} \mathcal{N}_0^2 = 0$$

$$\nabla \cdot \mathbf{u}_{ag} + \frac{\partial w_{ag}}{\partial z} = 0$$

"Residual-mean" equations:

$$\frac{\partial \overline{\mathbf{u}}_g}{\partial t} + \overline{\mathbf{u}}_g \cdot \nabla \overline{\mathbf{u}}_g + f_0 \mathbf{k} \times \tilde{\mathbf{u}}_{ag} + \beta y \, \mathbf{k} \times \overline{\mathbf{u}}_g + \frac{\nabla \overline{p}_{ag}}{\rho_0} = \boxed{-\mathbf{k} \times \overline{q' \mathbf{u}'}}$$

only eddy forcing

$$\frac{\partial \overline{b}}{\partial t} + \overline{\mathbf{u}}_g \cdot \nabla \overline{b} + \tilde{w}_{ag} \mathcal{N}_0^2 = 0$$

$$\nabla \cdot \tilde{\mathbf{u}}_{ag} + \frac{\partial \tilde{w}_{ag}}{\partial z} = 0$$

How to build momentum conservation into an eddy closure?

Write potential vorticity flux as:

"Taylor identity"

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

(Plumb 1986)

where: $M = \dfrac{\overline{v'^2 - u'^2}}{2}$ $\quad N = -\overline{u'v'}$ Reynolds stresses

$P = \dfrac{\overline{b'^2}}{2\mathcal{N}_0^2}$ eddy potential energy

$R = \dfrac{f_0}{\mathcal{N}_0^2}\overline{u'b'} \quad S = \dfrac{f_0}{\mathcal{N}_0^2}\overline{v'b'}$ eddy buoyancy flux (eddy form stress)

- these are the terms parameterised in Gent and McWilliams (1990)

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

Advantages:

1. Angular momentum constraints preserved if boundary conditions correctly applied.

2. If we neglect the Reynolds stresses, then reduces to parameterising eddy form stress as in Gent and McWilliams
⇒ natural framework for extending GM to include Reynolds stresses.

3. Second column is Eliassen-Palm flux (associated with propagation of wave activity).

4. Eddy energy provides an upper bound on a norm of the stress tensor:

$$\frac{1}{2}\left[(-N)^2 + (M-P)^2 + (M+P)^2 + N^2 + \frac{\mathcal{N}_0^2}{f_0^2}(R^2+S^2)\right] =$$
$$M^2 + N^2 + P^2 + \frac{\mathcal{N}_0^2}{2f_0^2}(R^2+S^2) \leq E^2$$

5. Energy conservation can be enforced via explicit eddy energy budget (cf. Eden and Greatbatch, 2008).

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

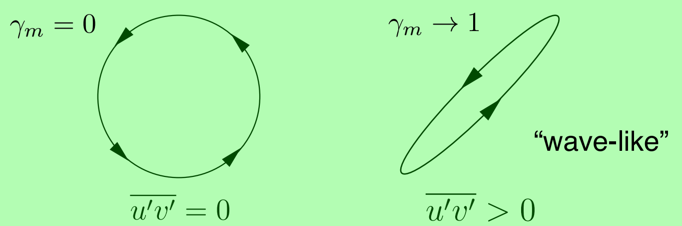
6. Energy norm allows eddy stress tensor to be rewritten, without loss of generality, in terms of eddy energy, 2 eddy anisotropies, and 3 eddy flux angles:

$$M = -\gamma_m E \cos 2\phi_m \cos^2\lambda \qquad N = \gamma_m E \sin 2\phi_m \cos^2\lambda \qquad P = E \sin^2\lambda$$

$$R = \gamma_b \frac{f_0}{\mathcal{N}_0} E \cos\phi_b \sin 2\lambda \qquad S = \gamma_b \frac{f_0}{\mathcal{N}_0} E \sin\phi_b \sin 2\lambda$$

horizontal orientation vertical orientation

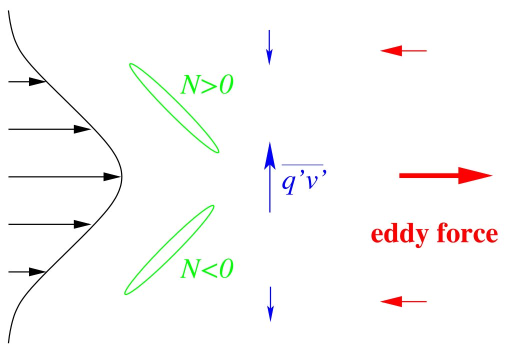
e.g., barotropic eddies:
(plan view)

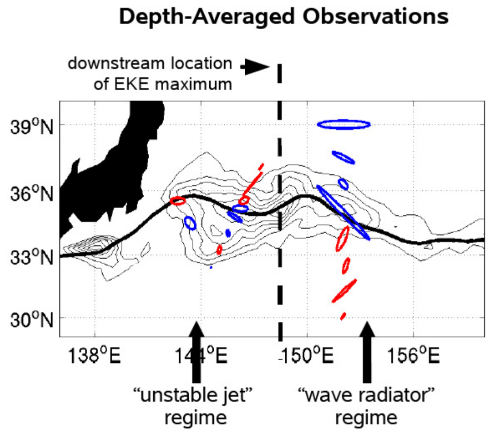


7. Unknowns are **nondimensional** and **≤ 1 in magnitude**.

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

8. Eddy flux angles have a strong connection with classical stability theory:
eddies lean "against" mean shear ⇒ extract energy from mean flow *(instability)*;
eddies lean "into" mean shear ⇒ return energy to mean flow *(stability)*.

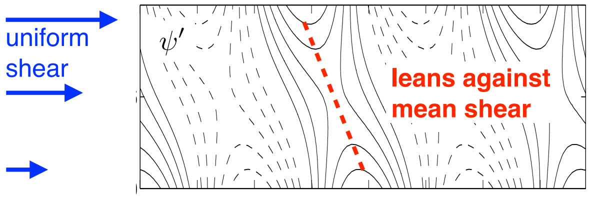




(Waterman et al. 2011)

Application to the Eady model

most **unstable** mode:



Eddy energy budget:

$$\frac{\partial}{\partial t}\int\!\!\int\!\!\int E\,dx\,dy\,dz = -\int\!\!\int\!\!\int \overline{u}\,\overline{q'v'}\,dx\,dy\,dz$$

$$= \int\!\!\int\!\!\int \frac{\partial \overline{u}}{\partial z} S\,dx\,dy\,dz$$

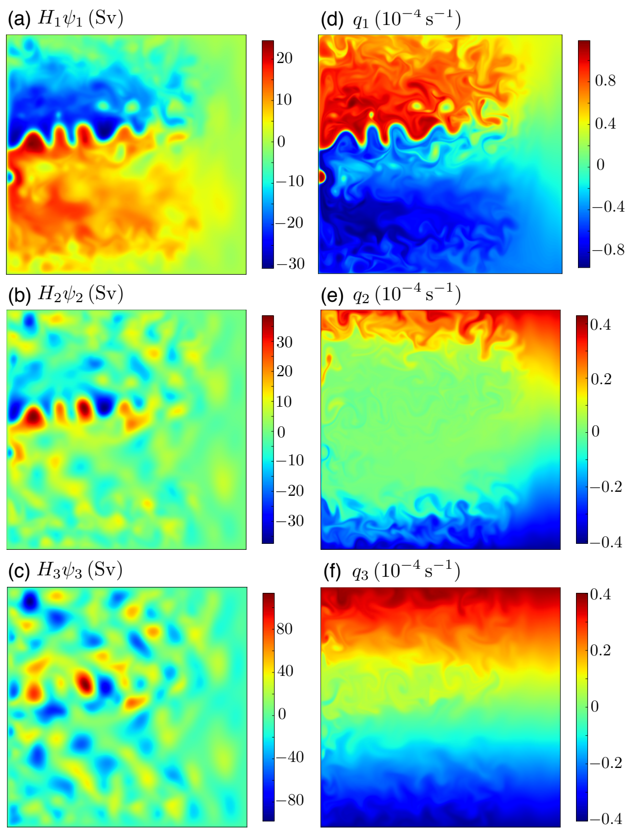
$$= \boxed{\alpha \frac{f_0}{N_0}\frac{\partial \overline{u}}{\partial z}} \int\!\!\int\!\!\int E\,dx\,dy\,dz$$

$\alpha \leq 1$

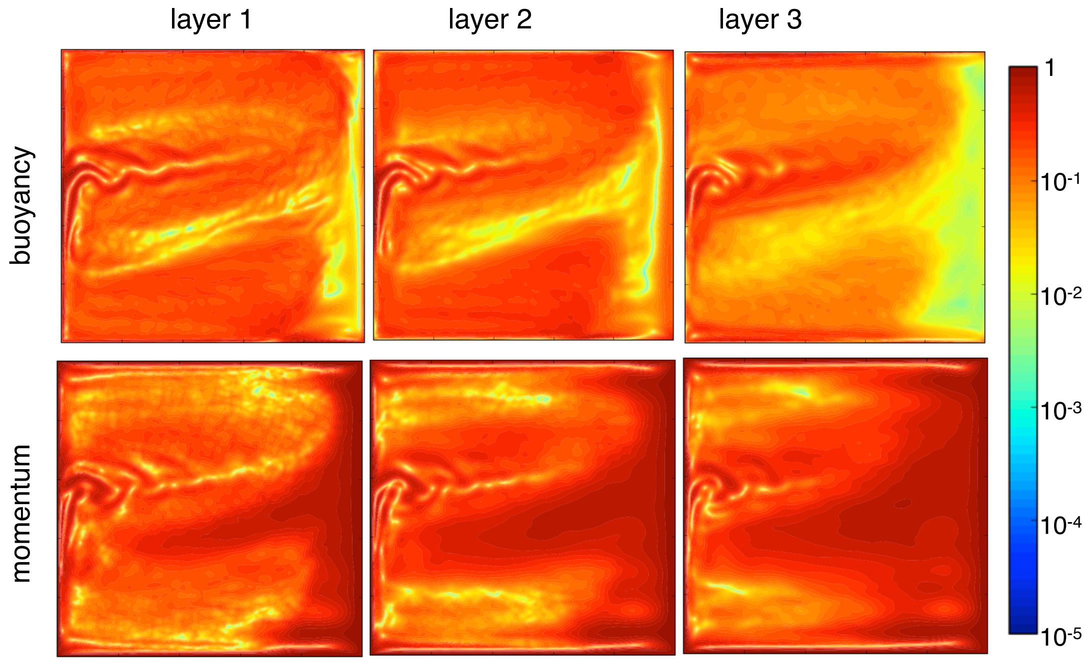
Eady growth rate
if $\alpha = 0.61$

can reverse argument to infer Gent and McWilliams diffusivity - turns out to be Visbeck et al. (1997)

Application to 3-layer, eddy-resolving quasigeostrophic basin model:

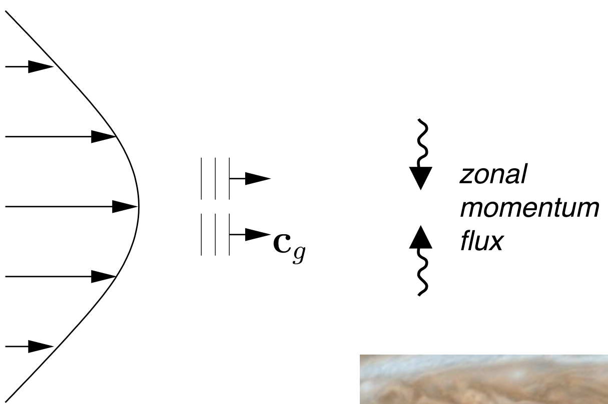


3-layer eddy-resolving basin model - eddy anisotropy



$$N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \qquad S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda$$

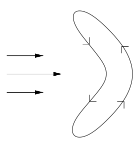
Eddy mixing angles: ray-tracing?



$$\dot{\mathbf{x}} = \frac{\partial \Omega}{\partial \mathbf{k}}, \qquad \dot{\mathbf{k}} = -\frac{\partial \Omega}{\partial \mathbf{x}}$$

$$\Omega = \hat{\Omega} + \mathbf{u} \cdot \mathbf{k} \approx -\frac{\beta k}{k^2 + l^2} + uk$$

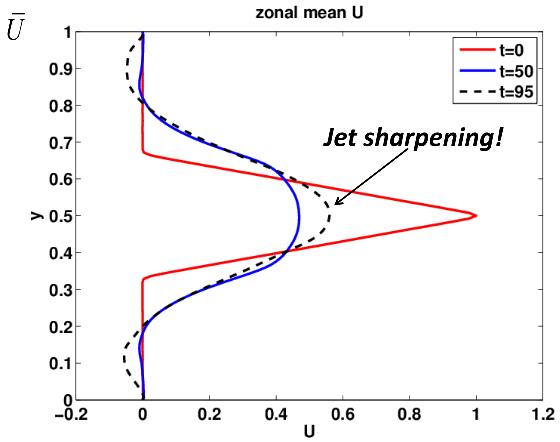
(e.g., Buhler and McIntryre 2005)

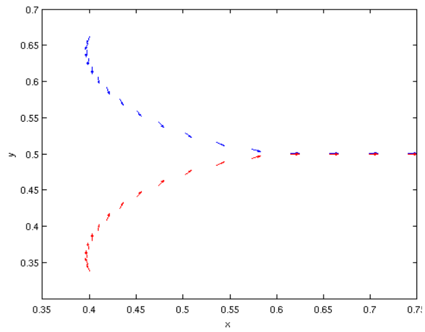


“banana-shaped” eddies ⇒ up-gradient momentum fluxes

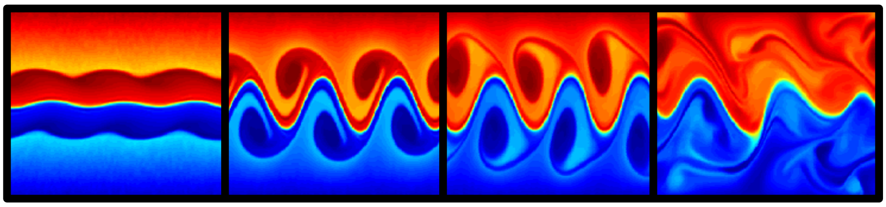
Simple ray-tracing pilot study (Talia Tamarin)

Piecewise linear barotropic jet with beta:





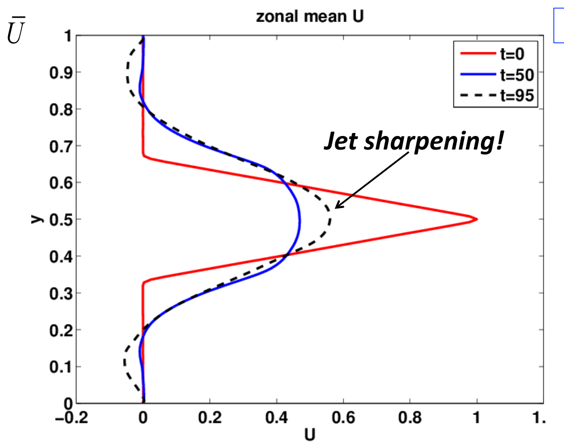
potential vorticity



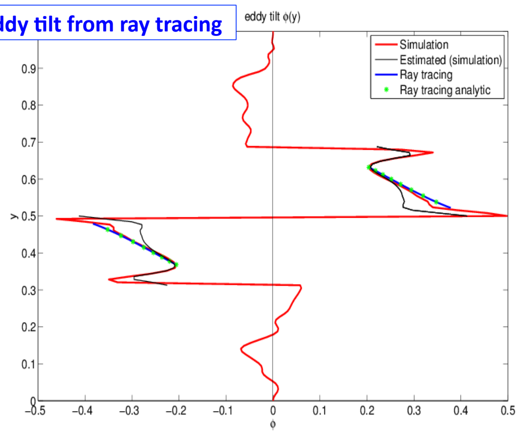
time →

Simple ray-tracing pilot study (Talia Tamarin)

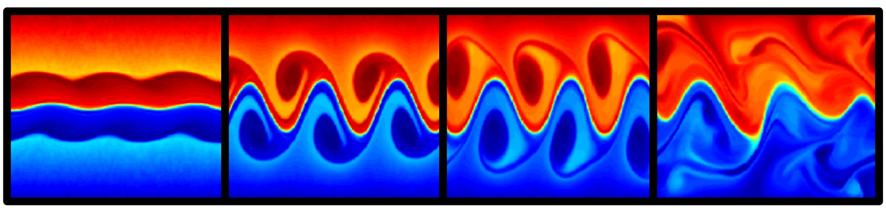
Piecewise linear barotropic jet with beta:



Eddy tilt from ray tracing



potential vorticity

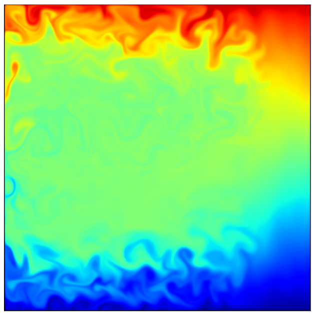


time →

What about mixing of potential vorticity?

If we: (i) solve an explicit eddy potential enstrophy ($\overline{q'^2}$) budget;
(ii) include dissipation of $\overline{q'^2}$ (= potential vorticity mixing);
(ii) ensure $\overline{q'\mathbf{u}'}$ vanishes when $\overline{q'^2}$ vanishes;
[use another bound on divergence of eddy stress tensor?]

then **Arnold's first stability theorem is preserved.**



Physical interpretation? (Marshall and Adcroft, 2010)

Eddy energy equation: $$\frac{\partial}{\partial t}\frac{\overline{\mathbf{u}'\cdot\mathbf{u}'}}{2} + \nabla\cdot(\ldots) = \overline{q'\mathbf{u}'}\cdot\nabla\psi$$

Eddy enstrophy equation: $$\frac{\partial}{\partial t}\frac{\overline{q'^2}}{2} + \nabla\cdot(\ldots) = -\overline{q'\mathbf{u}'}\cdot\nabla q$$

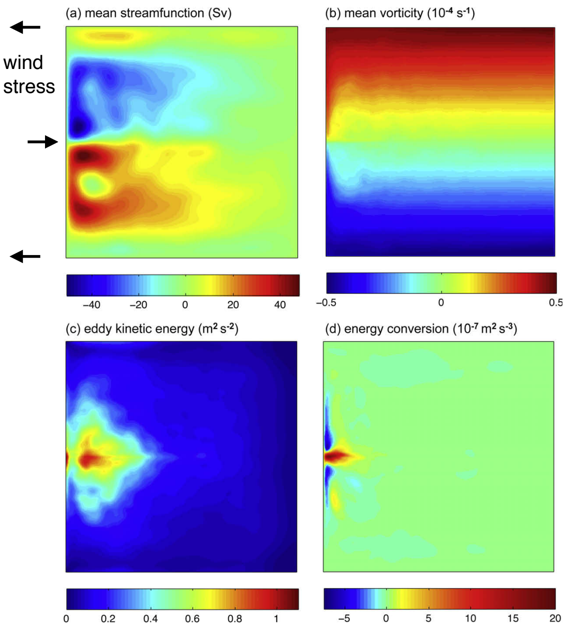
If $dq/d\psi > 0$, eddy energy can grow only at the expense of eddy potential enstrophy.

⇒ stable (in the sense of Lyapunov) - **Arnold's first stability theorem**.

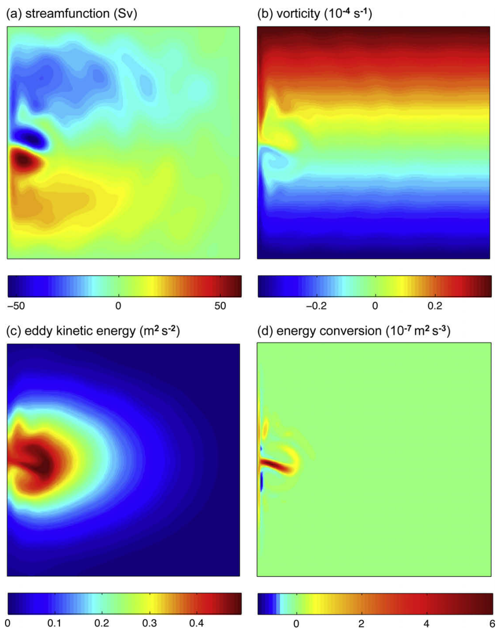
Qualitative illustration - *with energetically-consistent PV closure* (Marshall and Adcroft, 2010)

Wind-driven gyres (free-slip):

resolved eddies

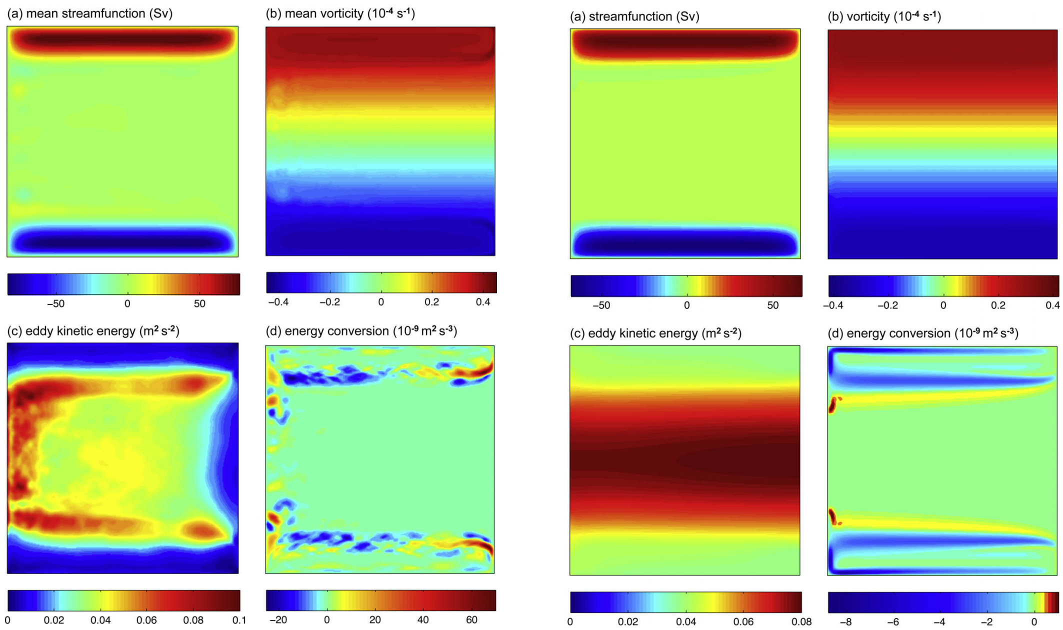


parameterised eddies

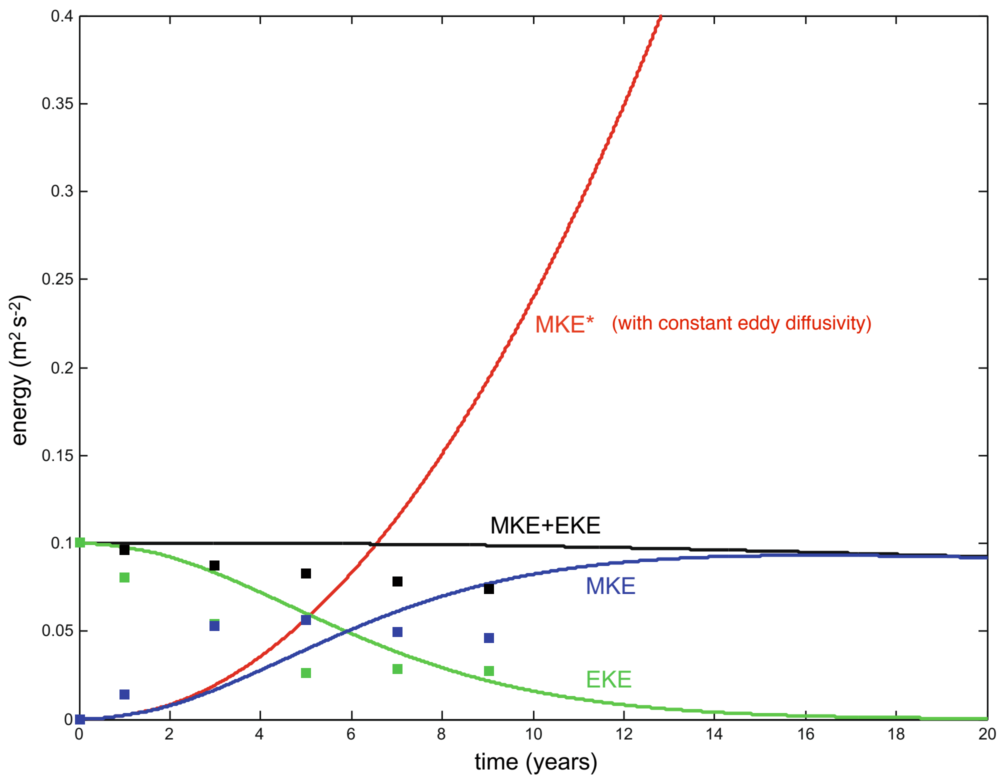


Freely-decaying turbulence (hyper-slip, initial uniform eddy energy):

resolved eddies — parameterised eddies



Freely-decaying turbulence - energetics:



More rigorous approach: coordinate-invariant derivation (Maddison and Marshall, 2013)

PV equation: $\partial_t \overline{q} + \left(\overline{[u_g]^a} \, \overline{q} \right)_{;a} = -T^{ab}_{;ab}$

double divergence
$\Rightarrow$ 2 forms of gauge freedom

eddy flux tensor

$$g_{ac} T^{cb} = \begin{pmatrix} N & M-K & R \\ M+K & -N & S \\ 0 & 0 & 0 \end{pmatrix}$$

"residual-mean"

$$\begin{pmatrix} N & M+K & 0 \\ M-K & -N & 0 \\ R & S & 0 \end{pmatrix}$$

Cronin (1996)

$$\begin{pmatrix} N & M-K & 0 \\ M+K & -N & 0 \\ R & S & 0 \end{pmatrix}$$

"half-residual mean"

$$\begin{pmatrix} N & M & \frac{1}{2}R \\ M & -N & \frac{1}{2}S \\ \frac{1}{2}R & \frac{1}{2}S & 0 \end{pmatrix}$$

Plumb (1986)

$$\begin{pmatrix} N & M-P & 0 \\ M+P & -N & 0 \\ R & S & 0 \end{pmatrix}$$

Hoskins et al. (1983) "E-vector"

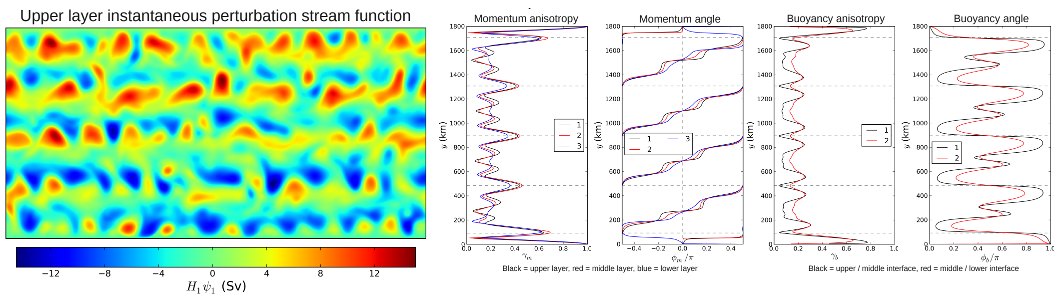
$$\begin{pmatrix} N & 2M & 0 \\ 0 & -N & 0 \\ R & S & 0 \end{pmatrix}$$

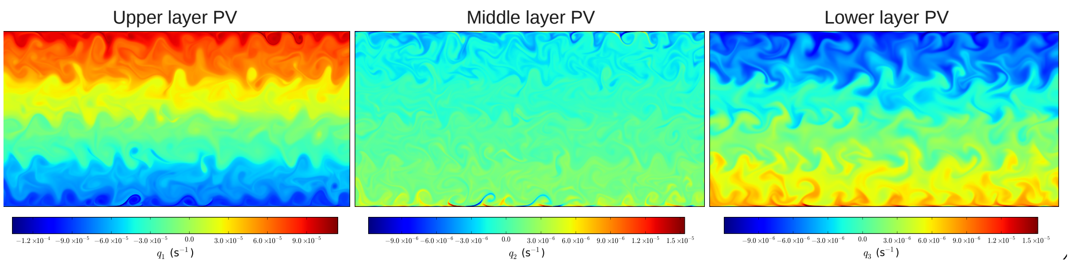
Approach generalises to isopycnal thickness-weighted primitive equations

(cf. Young, 2012)

Other work in progress

- Diagnosing the role of eddy fluxes in zonal jet formation





- Simple extension of Gent and McWilliams to include up-gradient momentum fluxes.
- Simple extension of Gent and McWilliams to include rectified eddy-topography interactions.

www.atm.ox.ac.uk/user/marshall

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A Framework for Parameterizing Eddy Potential Vorticity Fluxes

David P. Marshall and James R. Maddison
Department of Physics, University of Oxford, Oxford, United Kingdom

Pavel S. Berloff
Department of Mathematics, Imperial College of Science, Technology and Medicine, London, United Kingdom

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The Eliassen-Palm flux tensor

J. R. MADDISON[1]† AND D. P. MARSHALL[1]

Eddy Saturation of Equilibrated Circumpolar Currents

David R. Munday*
Department of Physics, University of Oxford, Oxford, UK

Helen L. Johnson
Department of Earth Sciences, University of Oxford, Oxford, UK

David P. Marshall
Department of Physics, University of Oxford, Oxford, UK

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Parameterization of ocean eddies: Potential vorticity mixing, energetics and Arnold's first stability theorem

David P. Marshall [a,*], Alistair J. Adcroft [b]
[a] Atmospheric, Oceanic and Planetary Physics, Clarendon Laboratory, Parks Road, Oxford OX1 3PU, United Kingdom
[b] Atmospheric and Oceanic Sciences Program, Princeton University, 300 Forrestal Road, Sayre Hall, Princeton, NJ 08544, United States

Summary of key points

• Geostrophic eddies play an key role in setting the mean structure of the ocean.

• Much of the success of the Gent and McWilliams eddy closure is down to preserving the near-adiabatic nature of the ocean interior.

• Preserving symmetries and conservation laws in models with parameterised eddies
$\Rightarrow$ classical stability conditions carry over:
- Eady growth rate;
- Charney-Stern;
- Arnold's first stability theorem.

• Down-gradient eddy potential vorticity flux closures are inconsistent with the underlying mathematical structure of the eddy-mean flow interaction.

• Gent and McWilliams is consistent with this underlying mathematical structure.

• Anisotropy ("wave-like" behaviour) is a prequesite for non-vanishing eddy fluxes.

• Much left to do!