Propagation, Stability and Instability of Small Amplitude Internal Waves

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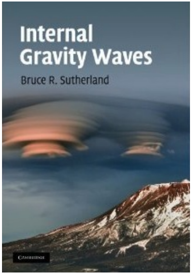


PDFs of lectures can be downloaded from
www.ualberta.ca/˜bsuther/talks/

Lectures are based in part upon textbook
"Internal Gravity Waves"
Cambridge University Press, 2010
ISBN-13: 978-0521839150

The Mathematical Treatment of Waves

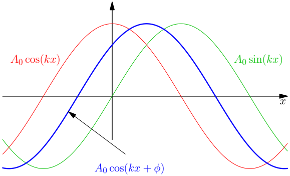
- plane waves, dispersion relation, phase velocity
- wavepackets, group speed
- dispersion, Schrödinger equation

Representation of Plane Waves

- Waves are characterized by changes some field, $\eta(\mathbf{x}, t)$, representing e.g. pressure, displacement, a velocity component, etc.
- Plane waves have infinite temporal and spatial extent and are periodic with a single (temporal) frequency ω and (spatial) frequency $\mathbf{k}$ - the "wavenumber vector".

- Plane waves of (half peak-to-peak) amplitude A_0 can be represented by

$$\begin{aligned} \eta(\mathbf{x}, t) &= A_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t) \\ \eta(\mathbf{x}, t) &= A_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \end{aligned}$$

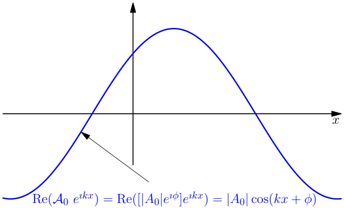


- Generally, plane waves are represented by

$$\eta(\mathbf{x}, t) = A_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t + \phi),$$

in which ϕ is the phase of the waves at the origin and $t = 0$.

Representation of Plane Waves



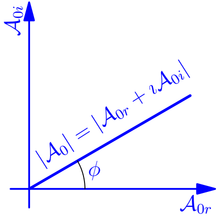
- In linear theory it is more convenient to incorporate the phase information into the amplitude through the complex representation of the waves:

$$\eta(\mathbf{x},t) = \mathcal{A}_0 \exp[\imath(\mathbf{k}\cdot\mathbf{x} - \omega t)],$$

  in which it is understood that the actual field is the real part of this expression.

- Here $\mathcal{A}_0 = \mathcal{A}_{0r} + \imath\mathcal{A}_{0i} = |\mathcal{A}_0|e^{\imath\phi}$ with

$$A_0 \equiv |\mathcal{A}_0| = \sqrt{{\mathcal{A}_{0r}}^2 + {\mathcal{A}_{0i}}^2} \text{ and } \phi = \tan^{-1}(\mathcal{A}_{0i}/\mathcal{A}_{0r})$$



- Generally, plane waves are best represented by

$$\eta(\mathbf{x},t) = \frac{1}{2}\mathcal{A}_0 \exp[\imath(\mathbf{k}\cdot\mathbf{x} - \omega t)] + c.c.$$

  in which $c.c.$ represents the complex conjugate of the preceding expression

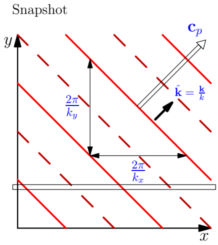
Phase Velocity

- The phase velocity is defined by

$$\mathbf{c}_p \equiv \frac{\omega}{|\mathbf{k}|}\hat{\mathbf{k}} = \frac{\omega}{|\mathbf{k}|^2}\mathbf{k}$$

- In two dimensions

$$(c_{px},\ c_{py}) = \left(\frac{\omega\ k_x}{{k_x}^2 + {k_y}^2},\ \frac{\omega\ k_y}{{k_x}^2 + {k_y}^2}\right)$$



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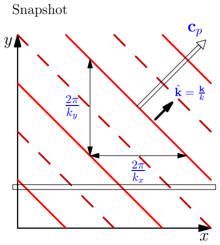
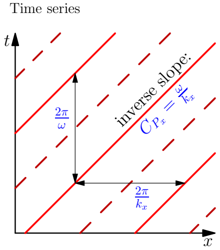
- The components of the phase velocity should not be confused with the speed determined from time series. For example, the speed of phase lines determined from a time series in x and t is

$$c_{Px} = \frac{\omega}{k_x}$$





Dispersion of Waves

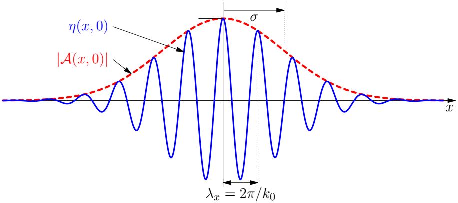
- Frequency, ω, is not independent of wavenumber, $\mathbf{k}$. In general, $\omega \equiv \omega(\mathbf{k})$, which is known as the "dispersion relation".

- Non-Dispersive Waves (eg sound, light, shallow water waves):
  - waves are *non-dispersive* if $\omega = ck$.
  - non-dispersive wavepackets do not change their shape.
  - their phase speed $c_p \equiv \omega/k = c$ is constant

- Dispersive Waves
  - most waves are *dispersive*.
  - eg deep water waves: $\omega = \sqrt{gk}$;   capillary waves: $\omega = \sqrt{\sigma k^3}$
  - typically dispersive waves spread as they propagate.

Wavepackets

- Represent wavepacket by $\eta(x,0) = \mathcal{A}(x,0)e^{\imath k_0 x}$
- If the wavepacket is "quasi-monochromatic", its envelope is much wider than its wavelength $\lambda = 2\pi/k_0$
  - *e.g.* $\mathcal{A}(x,0) = A_0 \exp(-x^2/2\sigma^2)$ with $\sigma \gg \lambda$



- For later times, write $\eta(x,t) = \mathcal{A}(x,t)e^{\imath(k_0 x - \omega(k_0)t)}$
- Write in terms of Fourier Transform:

$$\eta(x,0) = \int_{-\infty}^{\infty} \hat{\eta}(k)e^{\imath kx}\,dk \quad \Rightarrow \quad \eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k)e^{[\imath(kx-\omega t)]}\,dk$$

- For Gaussian wavepacket, $\hat{\eta}(k) = \frac{A_0}{\sqrt{\pi}\sigma}\exp[-(k-k_0)^2/(2/\sigma^2)]$

Group Speed

- We have written

$$\eta(x,t) = \mathcal{A}(x,t)e^{\imath(k_0 x - \omega(k_0)t)} = \int_{-\infty}^{\infty} \hat{\eta}(k)e^{[\imath(kx-\omega t)]}\,dk$$

Group Speed

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- Approximate $\omega \simeq \omega(k_0) + (k-k_0)\omega'(k_0)$.

$$\Rightarrow \quad \mathcal{A}(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\left[\imath(k-k_0)x - \imath\left\{(k-k_0)\omega'(k_0)\right\}t\right]}\, dk$$

- From this integral find that

$$\begin{aligned} \mathcal{A}_t &= -\imath\left\{(k-k_0)\omega'(k_0)\right\}\mathcal{A} \\ \mathcal{A}_x &= \imath(k-k_0)\mathcal{A} \end{aligned}$$

- Combine these to give $\quad \mathcal{A}_t + \omega'(k_0)\mathcal{A}_x = 0$
- This shows that the wavepacket translates at the group speed $c_g = \omega'(k_0)$.
- For non-dispersive waves ($\omega = ck$), the group speed equals the phase speed: $c_g = c_p = c$.

Theory for Linear Dispersion

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- Approximate $\omega \simeq \omega(k_0) + (k - k_0)\omega'(k_0) + \frac{1}{2}(k - k_0)^2\omega''(k_0)$.

$$\Rightarrow \quad \mathcal{A}(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\left[\imath(k-k_0)x - \imath\left\{(k-k_0)\omega'(k_0) + \frac{1}{2}(k-k_0)^2\omega''(k_0)\right\}t\right]}\, dk$$

- From this integral find that

$$\begin{aligned} \mathcal{A}_t &= -\imath\left\{(k - k_0)\omega'(k_0) + \frac{1}{2}(k - k_0)^2\omega''(k_0)\right\}\mathcal{A} \\ \mathcal{A}_x &= \imath(k - k_0)\mathcal{A} \\ \mathcal{A}_{xx} &= -(k - k_0)^2\mathcal{A} \end{aligned}$$

- Combine these to give $\imath(\mathcal{A}_t + \omega'(k_0)\mathcal{A}_x) + \frac{1}{2}\omega''(k_0)\mathcal{A}_{xx} = 0$
- Finally, transform to frame moving at speed $c_g \equiv \omega'(k_0)$ in which $X = x - c_g t$. This gives the Schrödinger Equation

$$\imath\mathcal{A}_t + \frac{1}{2}\omega''(k_0)\mathcal{A}_{XX} = 0$$

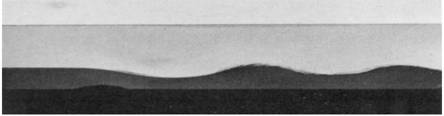
- This equation describes the dispersion of small amplitude wavepackets, i.e. the spreading of the amplitude envelope and the phase-shift of waves within it.

Interfacial and Internal Waves

- interfacial waves in 2-layer fluids, dead water
- internal waves, buoyancy frequency, momentum transport
- modes and geometric focusing of internal waves

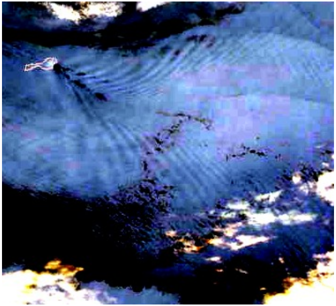
Interfacial Waves

- At the interface between light and dense fluid internal waves can be launched by flow over a hill.
- The wave crests can sometimes be made visible by clouds.



[From Fluid Mechanics Films, "Stratified Fluids", R. R. Long]

Internal waves in lee of an island



[uap-www.nrl.navy.mil/dynamics/html/mwfmjanmayen.html]

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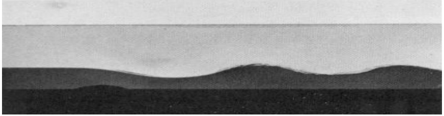

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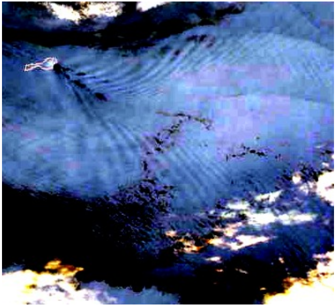
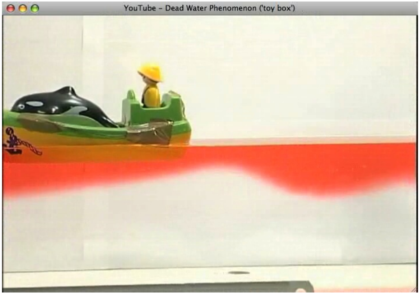


[uap-www.nrl.navy.mil/dynamics/html/mwfmjanmayen.html]

In the ocean, internal waves are generated when a boat travels in "dead water".



[From experiments by Vasseur, Mercier & Dauxois, (2010)]

Dispersion Relation of Interfacial Waves

- The corresponding dispersion relation is

$$\omega^2 = gk\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \simeq \frac{1}{2}g'k$$

in which the last expression invokes the Boussinesq approximation.

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in which the last expression invokes the Boussinesq approximation.

- More generally, for a two-layer fluid with depths H_1 and H_2, find

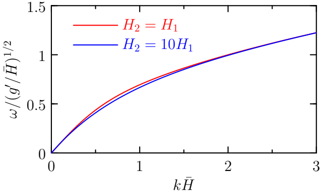
$$\omega^2 = g'k\frac{1}{\coth kH_1 + \coth kH_2}.$$

- In particular, for $kH_1 \ll 1$ and $kH_2 \ll 1$, get the shallow water dispersion relation:

$$\omega = \pm ck$$

in which

$$c = \sqrt{g'\bar{H}}, \quad \text{with} \quad \bar{H} = \frac{H_1 H_2}{H_1 + H_2}$$



- For shorter waves $c_g < c_p$: crests move from back to front of wavepacket.

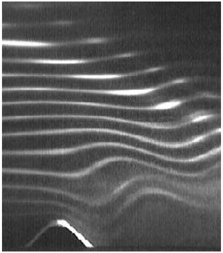
Internal Waves

- In a continuously stratified fluid lee waves move vertically behind the obstacle.

Lenticular Cloud over Mount Shasta, CA

[Photographed by Cindy Diaz, 2007, ShastaPhotografix.com]

Stratified flow over model hill



Private Comm., Hughes and Castro (1997)

Internal Waves

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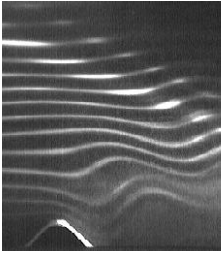
Lenticular Cloud over Mount Shasta, CA

[Photographed by Cindy Diaz, 2007, ShastaPhotografix.com]

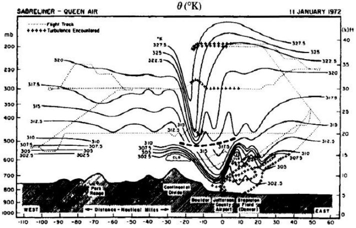
- The forcing can result in wave breaking close to the ground leading to chinook winds
- In extreme circumstances, downslope windstorms develop with hurricane-force warm, dry winds.

Stratified flow over model hill



Private Comm., Hughes and Castro (1997)

Potential temperature west of Denver



Observed Jan 1972 by Klemp and Lilly (1975)

The Buoyancy Frequency

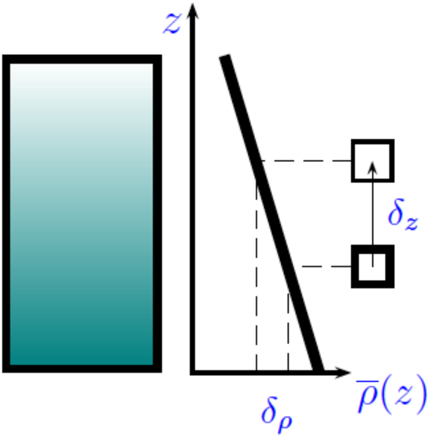
- Newton's Law ...

$$\rho \frac{d^2}{dt^2}\delta_z = -\delta_\rho \, g$$

- But $\delta_\rho \sim -\delta_z \frac{d\overline{\rho}}{dz}$

- That is ...

$$\frac{d^2}{dt^2}\delta_z + \left(-\frac{g}{\rho}\,\frac{d\overline{\rho}}{dz}\right)\delta_z = 0$$



This is spring equation with oscillation frequency

$$\omega \equiv N = \sqrt{-\frac{g}{\rho}\,\frac{d\overline{\rho}}{dz}}$$

Internal Waves: Oscillation Frequency

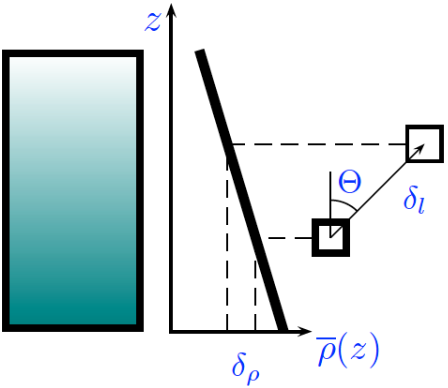
- Buoyancy force along l . . .

$$\rho \frac{d^2}{dt^2}\delta_l = -\delta_\rho \, g \cos\Theta$$

- But $\delta_\rho \sim -\delta_z \frac{d\overline{\rho}}{dz} \sim -\delta_l \frac{d\overline{\rho}}{dz} \cos\Theta$

- That is ...

$$\frac{d^2}{dt^2}\delta_l + \left(-\frac{g}{\rho}\frac{d\overline{\rho}}{dz}\right)\cos^2\Theta \, \delta_l = 0$$



This is spring equation with oscillation frequency

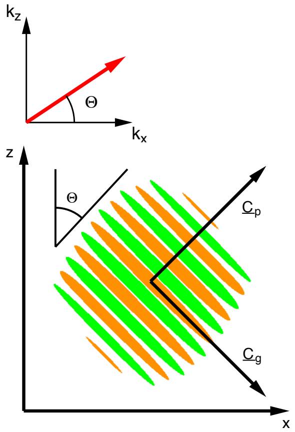
$$\boxed{\omega = N\cos\Theta}$$

Internal Wave Dispersion Relation

- The dispersion relation is (in 2D)

$$\omega^2 = N^2 \cos^2\Theta = N^2 \frac{k^2}{k^2+m^2}$$

- From this find
  - Phase Velocity:
    $$\underline{c}_p = \frac{\omega}{|\mathbf{k}|}\hat{k} \;\propto\; (k, m)$$
  - Group Velocity:
    $$\underline{c}_g = \nabla_{\mathbf{k}}\,\omega \propto (m, -k)$$



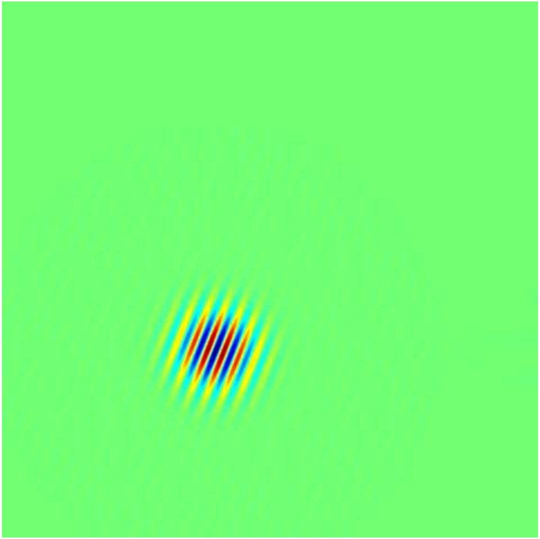
- Note $\underline{c}_p \perp \underline{c}_g$: Phase goes up, waves go down

Wavepacket Propagation

Simulation of a Small-Amplitude Vertically Propagating Wavepacket

$$m = -0.4k$$

Note how the wavepacket moves upward to the right while the crests move downward to the right.



Simulation of Oscillating Cylinder

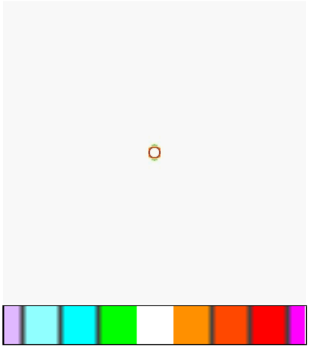
- The movie will show vertical displacement field, ξ, as it evolves over time.
- The circular region is periodically forced with frequency,

$$\omega = N/\sqrt{2}$$

- So expect

$$\boxed{\Theta = \cos^{-1}(1/\sqrt{2}) = 45^o}$$

$\xi(x, z, t)$



$\xi < 0$ $\xi > 0$

Also see talks by Staquet and Peacock on tidal flow.

Internal Wave Polarization Relations

- From the conservation equations, can determine relationship between different fields in terms of the vertical displacement,

$$\xi = \text{Re}\left(Ae^{\imath(kx+mz-\omega t)}\right) = A\cos(kx+mz-\omega t).$$

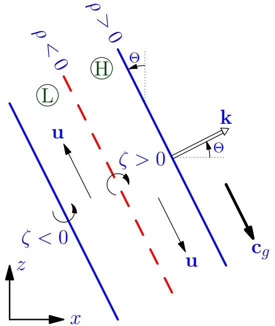
- For example, from $w = \xi_t$, the vertical velocity is

$$w = A\omega\sin(kx+mz-\omega t)$$

From $u_x + w_z = 0$, the horizontal velocity is

$$u = -A\omega\frac{m}{k}\sin(kx+mz-\omega t)$$

Note that u and w are in phase.



- Likewise find that pressure, p, is in phase with $\mathbf{u}$
- The spanwise vorticity component, ζ, is in phase with ξ.

Evanescent Internal Waves

- If a stratified fluid is forced at a (supercritical) frequency, ω, greater than the buoyancy frequency, N, then evanescent disturbances are created.
- To see this, rearrange the dispersion relation:

$$\omega^2 = N^2 \frac{k^2}{k^2 + m^2} \quad \Rightarrow \quad m^2 = k^2 \left(\frac{N^2}{\omega^2} - 1 \right).$$

The right-hand side is negative if $\omega > N$. So $m = \pm \imath \gamma$ with

$$\gamma = k \sqrt{1 - \frac{N^2}{\omega^2}}.$$

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The right-hand side is negative if $\omega > N$. So $m = \pm \imath \gamma$ with

$$\gamma = k \sqrt{1 - \frac{N^2}{\omega^2}}.$$

- Thus, for bounded solutions above a source of fast forcing

$$\xi = \text{Re} \left(A e^{\imath(kx + mz - \omega t)} \right) = A e^{-\gamma z} \cos(kx - \omega t).$$

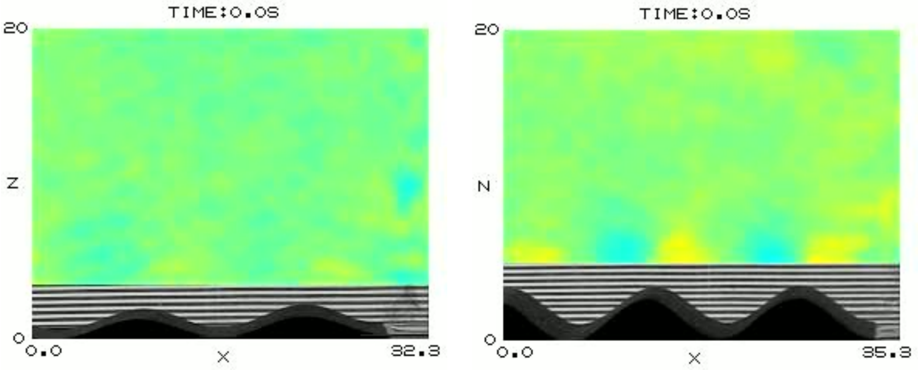
- In this case, u and w are 90 degrees out of phase:

$$\begin{aligned} w &= A\omega e^{-\gamma z} \sin(kx - \omega t) \\ u &= -A\omega \frac{\gamma}{k} e^{-\gamma z} \cos(kx - \omega t) \end{aligned}$$

Uniformly Stratified Flow over Mountains

- A model set of hills is towed below a stratified fluid.
- Schlieren is used to visualize the waves. The computed crests (warm colours) and troughs (cold colours) are shown.

Small Sinusoidal Hills; Subcritical — *Large Sinusoidal Hills; Supercritical*



[Aguilar, BRS & Muraki, Deep-Sea Res. (2006)]

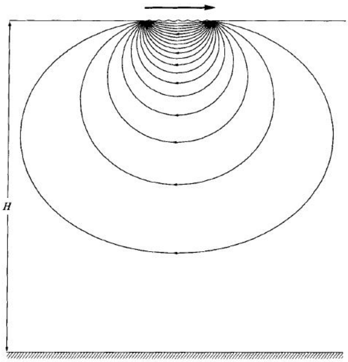
Momentum Transport and Deposition

- When waves break they deposit momentum to the mean flow.
- Even before they break, a wave-induced mean flow is associated with waves
  - for surface waves, this is the "Stokes drift" . . . a.k.a. the wave-induced mean flow

- Like surface waves, internal waves induce a mean flow
- Unlike surface waves, the wave-induced mean flow of internal waves has the most important influence upon the evolution of finite-amplitude wavepackets

The (Surface) Wave Momentum Myth

- From McIntyre's (JFM 1981) paper "On the 'Wave Momentum' Myth":
  - waves in media transport momentum but do not in themselves possess momentum
  - eg a deep return flow ensures zero vertically-integrated momentum
  - so fluid does not "pile up" at leading edge of a *wavepacket*



[McIntyre, J. Fluid Mech. (1981)]

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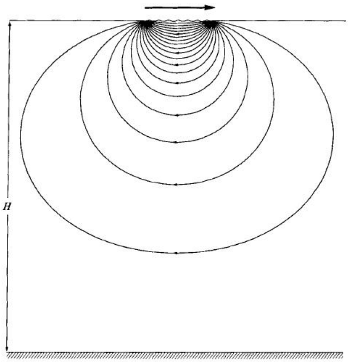
  - so fluid does not "pile up" at leading edge of a *wavepacket*

- But fluid does move slowly in the wave direction: the "Stokes drift"
- The Stokes drift can be derived from the flux-form of the horizontal momentum equation:

$$\begin{aligned}\partial_t U &= -\partial_x \langle u\, u\rangle - \partial_z \langle uw\rangle - \partial_x \langle p/\rho_0\rangle \\ &\rightarrow -\partial_x \langle uu\rangle\end{aligned}$$

with
$u = A_u\left(\epsilon(x - c_{gx}t), \epsilon^2 t\right)\ e^{kz}\ e^{\imath(kx-\omega t)}$.

$$\Rightarrow U = \frac{1}{c_{gx}}\langle u\, u\rangle \quad = \quad \frac{1}{2c_{gx}}|A_u|^2 e^{2kz}$$

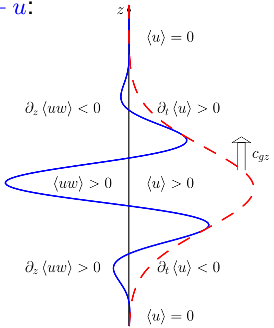


[McIntyre, J. Fluid Mech. (1981)]

The Internal Wave Momentum Fact

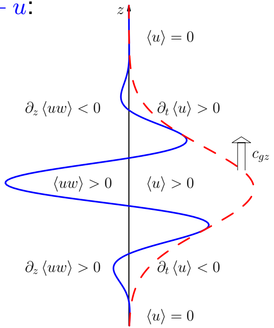
- Flux-form of x-momentum eqn: $\frac{\partial u}{\partial t} + \frac{\partial\ uu}{\partial x} + \frac{\partial\ uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial\ p}{\partial x} \qquad (\star)$
- Suppose u includes a background flow so that $u \to U + u$:
  - $U(z,t)$ is the mean horizontal (wave-induced) flow
  - $u(x,z,t)$ is the perturbed flow due to waves
- Substitute into $(\star)$ and average using $\langle u \rangle = \langle w \rangle = 0$:

$$\frac{\partial U}{\partial t} = -\frac{\partial\ \langle uw \rangle}{\partial z}$$



The Internal Wave Momentum Fact

- Flux-form of x-momentum eqn: $\frac{\partial u}{\partial t} + \frac{\partial\ uu}{\partial x} + \frac{\partial\ uw}{\partial z} = -\frac{1}{\rho_0}\frac{\partial\ p}{\partial x}$ $(\star)$
- Suppose u includes a background flow so that $u \to U + u$:
  - $U(z,t)$ is the mean horizontal (wave-induced) flow
  - $u(x,z,t)$ is the perturbed flow due to waves



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- Assume $u = A_u(Z,T)\, e^{\imath(kx+mz-\omega t)}$, etc
  with $Z = \epsilon(z - c_{gz}t)$ and $T = \epsilon^2 t$.

  Hence $U \ = \ \frac{1}{c_{gz}} \langle uw \rangle \ = \ \frac{1}{2} N|\mathbf{k}||A_\xi|^2$

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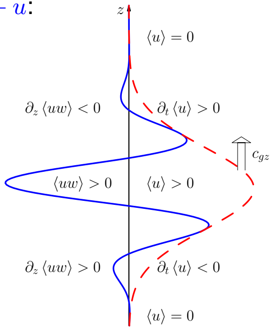

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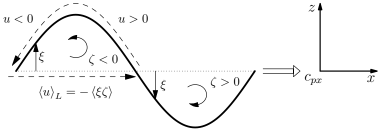
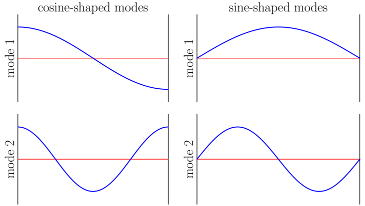
- Using the polarization relations, find that this equals the Kelvin Impulse $-\langle\xi\zeta\rangle$,
  in which ξ is the vertical displacement and ζ is the spanwise vorticity.



Wave Modes

- In a finite-size domain, waves whose horizontal extents are comparable to the domain size are best described by modes.



- Effectively, this amounts to constructing Fourier sine or cosine series in a domain of length L rather than a Fourier transform:
  - With Dirichlet (zero value, eg no-slip) boundary conditions,

$$\eta(x,t) = \sum_{n=1}^{\infty} \mathcal{A}_n \sin(k_n x) e^{-\imath\omega t}, \quad \text{in which} \quad k_n = \pi/L.$$

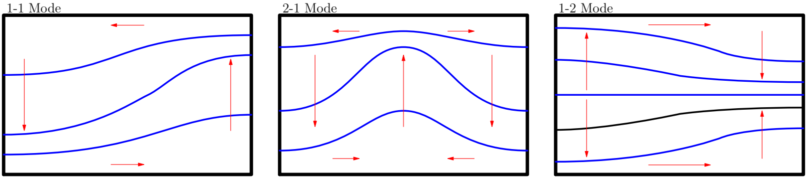
  - With Neumann (zero slope, eg free-slip) boundary conditions,

$$\eta(x,t) = \sum_{n=1}^{\infty} \mathcal{A}_n \cos(k_n x) e^{-\imath\omega t}.$$

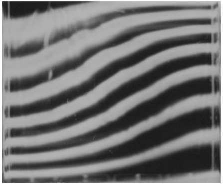
- The frequency of each mode is still given by the dispersion relation, $\omega = \omega(k)$, detemined for plane waves, except that only a discrete set of values, $\omega_n = \omega(k_n)$, are realized.

Wave Modes in 2D

- Domain-size internal waves in a rectangular box can be described by low-order modes in the horizontal and vertical.



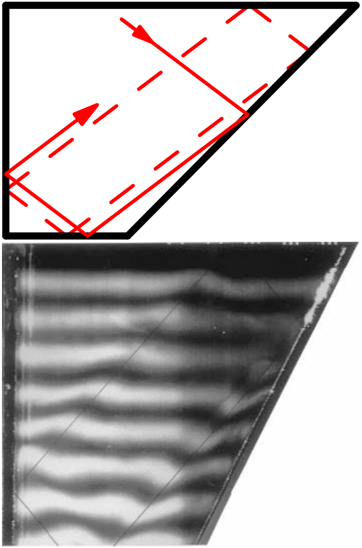
- Such modes can be generated, for example, by putting a tank of salt-stratified fluid on a table that oscillates at the frequency of one of the modes.



Benielli and Sommeria, J. Fluid Mech., 1998

Internal Wave Attractors

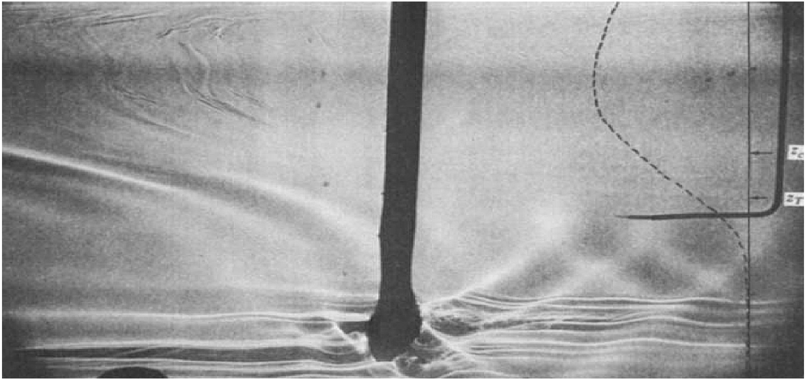
- If the tank has sloping walls, then oscillating the tank at a fixed frequency does not necessarily excite tank-scale modes.
- Instead, disturbances form along a parallelogram in which each side forms and angle $\Theta = \cos^{-1}(\omega/N)$ with the vertical.
- This is known as an "attractor". Its shape can be predicted by imagining a beam at a fixed angle repeatedly reflecting from the boundaries of the tank.



Maas and Benielli, Nature, 1997

Waves in Non-uniform Media

- ray theory (aka “WKB(J) theory”)
- critical and reflection levels, tunnelling
- WKB renormalization (aka WKB stretching)



Ray Theory … aka WKB(J) theory

- Represent waves by $\psi(\mathbf{x},t) = \Psi(\mathbf{x},t)e^{\imath\alpha(\mathbf{x},t)}$ in which Ψ is a slowly varying amplitude envelope and $\alpha \sim \mathbf{k}\cdot\mathbf{x} - \omega t.$ So $\nabla\alpha = \mathbf{k}$, $\frac{\partial\alpha}{\partial t} = -\omega$
  - Eliminating α gives
$$\frac{dk_i}{dt} = -\frac{\partial\omega}{\partial x_i}$$
  - In frame moving at group velocity, $\underline{c}_g = \nabla_{\mathbf{k}}\,\omega$:
$$\frac{dx_i}{dt} = \frac{\partial\omega}{\partial k_i}$$
  - These 6 coupled ODEs are Hamiltonian equations

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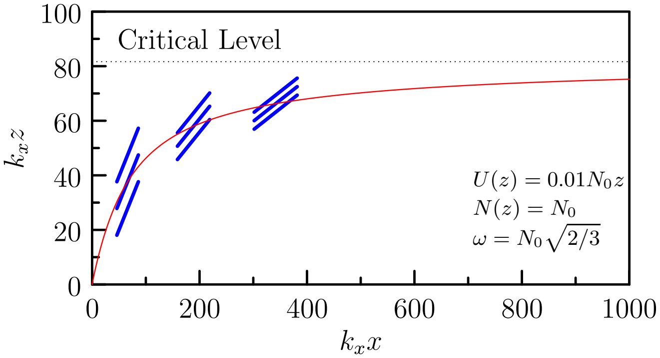
  - These 6 coupled ODEs are Hamiltonian equations
- In particular, for a time-independent system, $\omega = \omega(\mathbf{k},\mathbf{x})$.
  - hence $\frac{d\omega}{dt} = \nabla_{\mathbf{k}}\omega\cdot\frac{d\mathbf{k}}{dt} + \nabla\omega\cdot\frac{d\mathbf{x}}{dt} = 0.$
  - so the intrinsic frequency, ω, is constant during motion.
- Likewise, for x-invariant system, k_x is constant during motion.

Ray Equations in 2D Parallel Flow

- Assume background flow is $U(z) \;\Rightarrow\; \left(\frac{dx}{dt}, \frac{dz}{dt}\right) = (U(z) + c_{gx}, c_{gz})$
- So path of wavepacket is $z(x)$ satisfying
$$\frac{dz}{dx} = \frac{c_{gz}}{U(z) + c_{gx}}.$$
- Invariance in time $\;\Rightarrow\; \omega$ is constant
- Invariance in $x \;\Rightarrow\; k$ is constant
- Doppler-shifted frequency $\Omega \equiv \omega - kU(z)$ varies with z.
- Vertical wavenumber and group velocity can be found from the dispersion relation, eg
$$\Omega(z) = N(z)\frac{k}{\sqrt{k^2 + m^2}} \;\Rightarrow\; m = \pm k\sqrt{\frac{N^2}{\Omega^2} - 1}$$

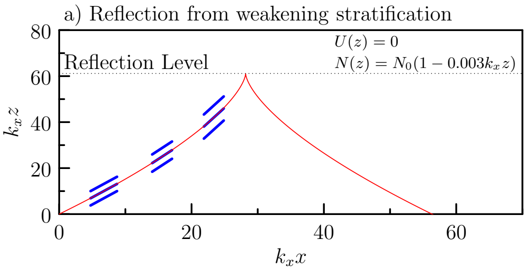
Internal Waves Approaching Critical Levels

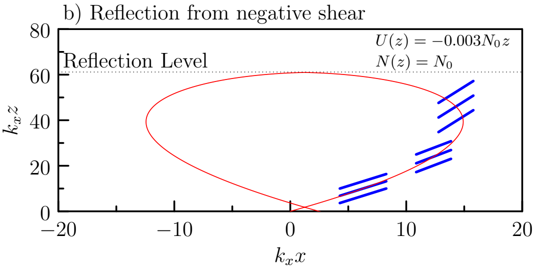
- Using ray theory (aka WKB theory) one can estimate the path followed by a small-amplitude quasi-monochromatic wavepacket in non-uniformly stratified fluid with non-uniform background flow.



Internal Waves Approaching Reflection Levels

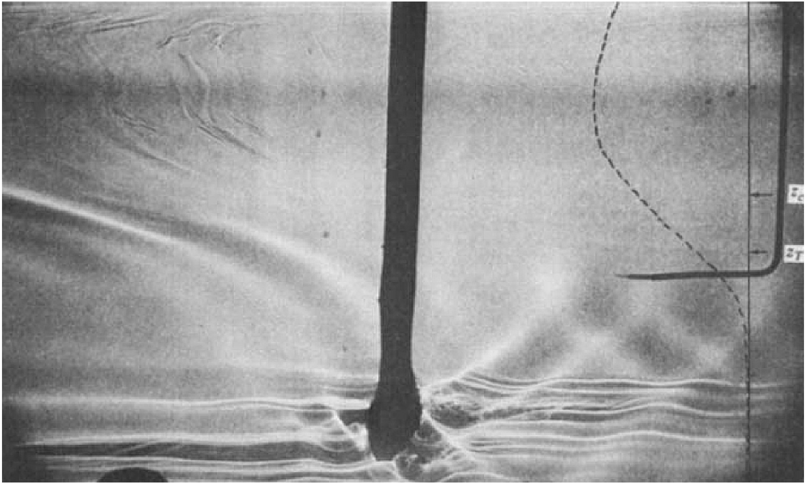
- Internal waves can reflect . . .
  - where the stratification becomes weak
  - where the background wind blows against the waves increasing their Doppler-shifted frequency





Critical and Reflection Levels

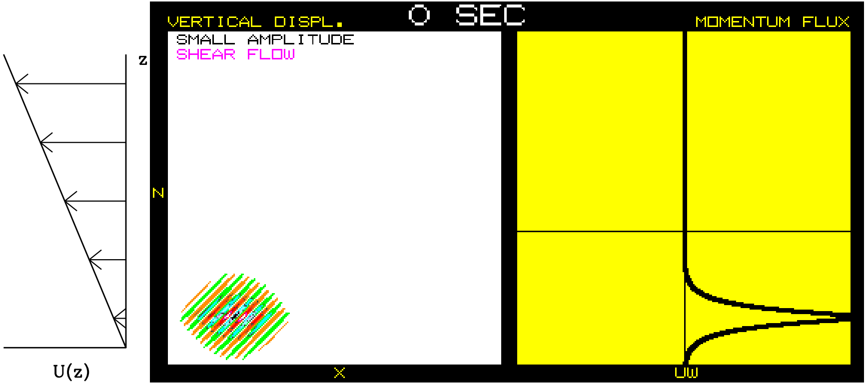
Internal waves approaching critical and reflection levels observed in a laboratory experiment



[Koop, J. Fluid Mech. (1981)]

Wavepacket Propagation in Shear

Simulation of Wavepacket in Uniform Negative Shear



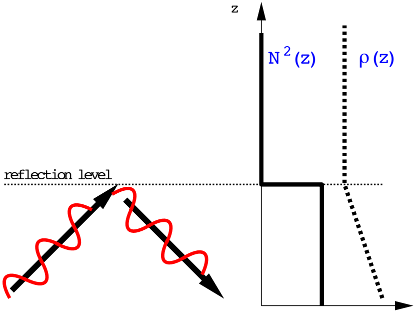
$A \equiv \frac{A_\xi}{\lambda_x} \simeq 0.0012,\ k_z = -0.7 k_x$

Vertical Momentum Transport

[BRS, Quart. J. Roy. Met. Soc. (2000)]

Reflection and Tunnelling

- A heuristic application of ray theory treats waves like particles: wavepackets "bounce off" constant density regions.



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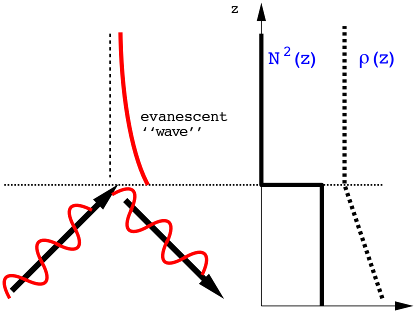

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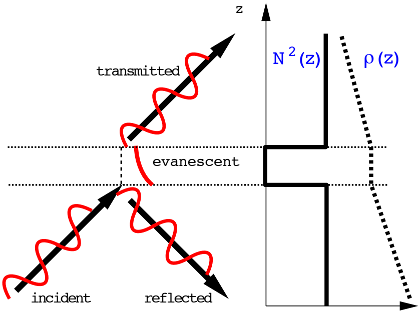
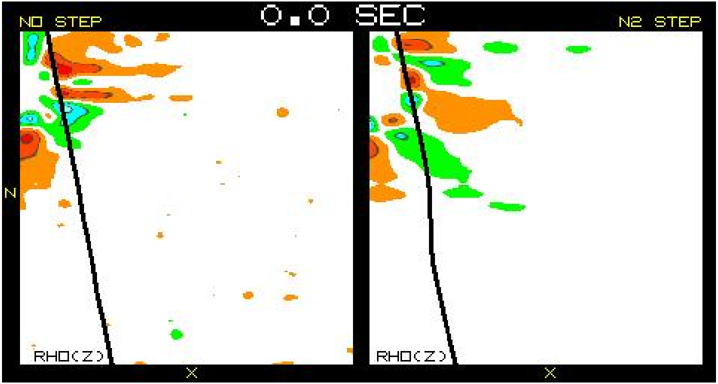


- If the constant-density region is small, waves reappear on the other side.
  This can be observed in the laboratory ...

Observation of Tunnelling

Waves in uniform and non-uniform stratification



Uniform stratification
(constant density gradient)

Variable stratification
(mixed region at mid-depth)

[BRS & Yewchuk, J. Fluid Mech. (2004); Gregory & BRS, Phys. Fluids (2010)]

See also Mathur & Peacock, J. Fluid Mech. (2009); Phys. Rev. Lett. (2010)

WKB Stretching

- WKB theory also determines the structure of modes in non-uniform media.
- For example, in non-uniform stratification the streamfunction is $\psi(x,z,t) = \hat{\psi}(z)\,\exp[\imath(kx - \omega t)]$ in which

$$\frac{d^2\hat{\psi}}{dz^2} + m^2\hat{\psi} = 0, \quad \text{in which} \quad m^2(z) = k^2\left[\frac{N^2(z)}{\omega^2} - 1\right].$$

- Assuming the vertical variations of $\hat{\psi}$ are small compared to those of N, we find

$$\hat{\psi} \propto m^{-1/2}\exp\left(\pm\imath\int m\,dz\right) \simeq \sqrt{\frac{N_\star}{N}}\,\sin(m_j Z),$$

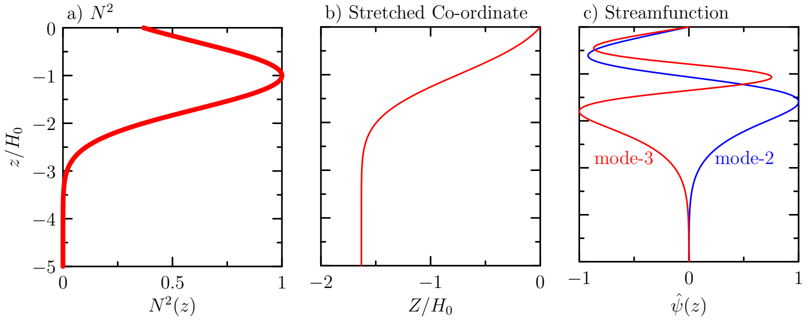
in which (assuming hydrostatic so $m \propto N$)

$$Z \equiv \frac{1}{N_\star}\int_z^0 N(\tilde{z})\,d\tilde{z}, \;\; m_j = j\pi N_\star\left[\int_{-H}^0 N\,dz\right]^{-1}, \; j = 1, 2, \ldots$$

and $N_\star$ is a characteristic value of N.

WKB Stretching (cont'd)

- For example, consider high-order modes in a model thermocline:



WKB Stretching (cont'd)

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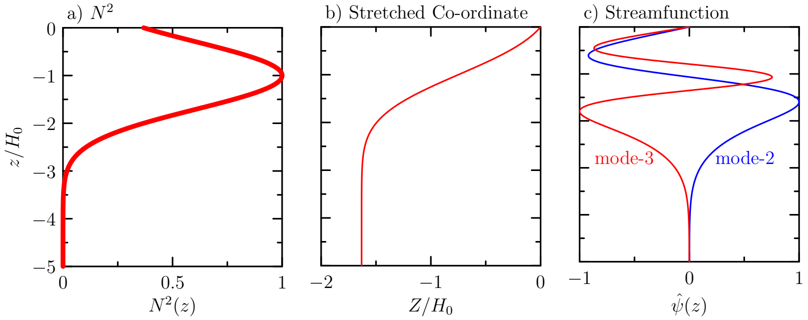
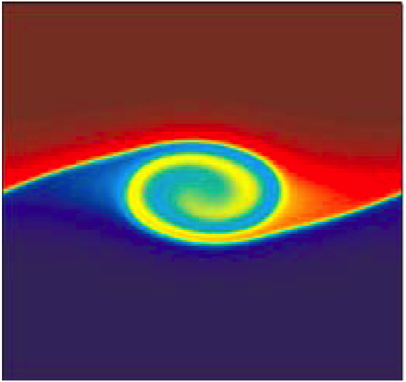
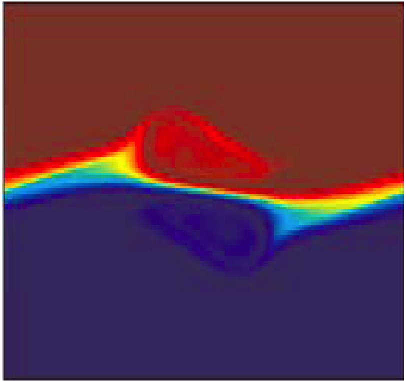


- *A NOTE OF CAUTION!*
  - WKB theory is a good approximation only if the background varies slowly compared with the vertical structure of the waves.
  - It is questionably accurate when considering examining mode-1 waves . . . internal waves are very different from interfacial waves.

Linear Stability Theory

- Taylor-Goldstein equation
- Kelvin-Helmholtz, Holmboe and Taylor Instability
- spatial, absolute/convective instability, parametric subharmonic instability





Stability of Stratified Shear Flow

- Consider non-uniformly stratified fluid with buoyancy frequency, $N(z)$ and background horizontal flow $\bar{U}(z)$.
- The linearized equations of motion are

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \bar{U}\frac{\partial u}{\partial x} + w\frac{d\bar{U}}{dz} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
\frac{\partial w}{\partial t} + \bar{U}\frac{\partial w}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{1}{\rho_0}\rho g, \\
\frac{\partial \rho}{\partial t} + \bar{U}\frac{\partial \rho}{\partial x} &= w\frac{\rho_0}{g}N^2, \\
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}
$$

- Looking for periodic solutions of the form $b(x,z,t) = \hat{b}(z)e^{\imath(kx-\omega t)}$, can combine to form a single equation for the streamfunction amplitude $\hat{\psi}(z)$:

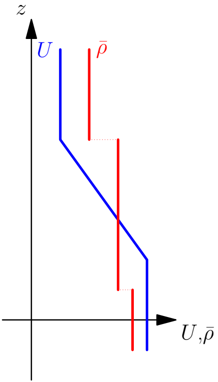
$$
\frac{d^2\hat{\psi}}{dz^2} + \left[\frac{N^2}{(\bar{U}-c)^2} - \frac{\bar{U}''}{(\bar{U}-c)} - k^2\right]\hat{\psi} = 0,
$$

in which $c \equiv \omega/k$. This is the *Taylor-Goldstein equation*.

Piecewise-Linear Flows

- Insight into the stability of stratified shear flows is gained by examining background flows that are piecewise linear and background density profiles that are piecewise constant.
- If $N = 0$ where $U = U_0 + sz$ (constant shear), the Taylor-Goldstein equation reduces to

$$\frac{d^2\hat{\psi}}{dz^2} - k^2\hat{\psi} = 0$$



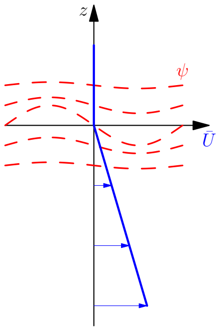
- Thus we can find analytical solutions for flows that are piecewise-linear insisting upon continuity of mass and pressure:

$$\frac{\hat{\psi}}{\bar{U} - c} \quad \text{and} \quad \bar{\rho}\left[(\bar{U} - c)\hat{\psi}' - \bar{U}'\hat{\psi} - \frac{g}{\bar{U} - c}\hat{\psi}\right].$$

Shear-Rayleigh Waves

- Consider uniform-density fluid with a "kinked-shear" horizontal velocity profile:

$$\bar{U} = \begin{cases} 0 & z \geq 0 \\ -s_0 z & z < 0. \end{cases}$$



- Requiring bounded solutions, the streamfunction amplitude is

$$\hat{\psi} = \begin{cases} \mathcal{A}e^{-kz} & z \geq 0 \\ \mathcal{B}e^{kz} & z < 0. \end{cases}$$

- Applying the interface conditions at $z = 0$ gives the matrix eigenvalue problem

$$\begin{pmatrix} 1 & -1 \\ ck & ck - s_0 \end{pmatrix} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

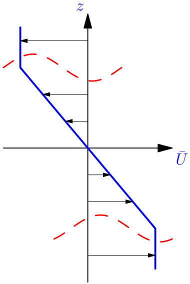
in which $c = \omega/k$.

- For nontrivial solutions, insist $2ck - s_0 = 0$. Thus we have found the dispersion relation for "shear-Rayleigh" waves: $\omega = s_0/2$.

Finite-Depth Shear Layer

- Consider uniform-density fluid with two kinks in the velocity profile:

$$\bar{U} = \begin{cases} -U_0 & z \geq H \\ -U_0 \frac{z}{H} & |z| < H \\ U_0 & z \leq -H \end{cases}$$



- Requiring bounded solutions, find

$$\hat{\psi} = \begin{cases} \mathcal{A} e^{-kz} & z \geq H \\ \mathcal{B}_1 \sinh kz + \mathcal{B}_2 \cosh kz & |z| < H \\ \mathcal{C} e^{kz} & z \leq -H \end{cases} .$$

- Now apply the interface conditions at $z = H$ and $z = -H$. This gives a 4x4 matrix eigenvalue problem from which we find the dispersion relation:

$$\omega^2 = \frac{1}{4} {s_0}^2 \left[(1 - 2kH)^2 - e^{-4kH} \right], \quad \text{in which} \quad s_0 = U_0/H.$$

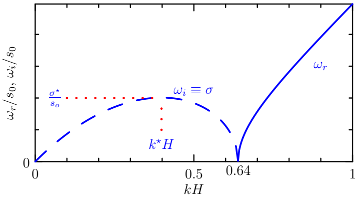
- Note, as $kH \to \infty$, $\omega \to \pm(s_0/2 - U_0 k)$. Each root corresponds to a shear-Rayleigh waves at the upper and lower kink in the shear.

Kelvin-Helmholtz Instability

- The right-hand side of the dispersion relation

$$\omega^2 = \frac{1}{4}{s_0}^2\left[(1-2kH)^2 - e^{-4kH}\right],$$

is negative if $0 < kH \lesssim 0.64$. Hence ω is imaginary: $\omega = \pm\imath\sigma$.



- Generally, if $\omega = \omega_r + \imath\sigma$,

$$\psi(x,z,t) = \hat{\psi}(z)e^{\imath(kx-\omega t)} = \hat{\psi}(z)e^{\imath(kx-\omega_r t)}e^{\sigma t}.$$

So the disturbance grows exponentially in time with growth rate σ.

- The largest growth rate is $\sigma^\star \simeq 0.20 s_0$, for which $k^\star \simeq 0.40/H$.
  - This gives the wavelength of disturbances most likely to grow from a uniform-density shear layer of half-depth, H.

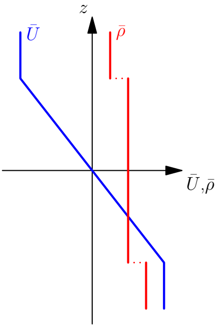
http://www.cambridge.org/features/worsterMovies/

Kelvin-Helmholtz Instability in Stratified Fluid

UNIVERSITY OF ALBERTA

- Now consider the shear layer in a (symmetric) 3-layer fluid with background density profile

$$\bar{\rho} = \begin{cases} \rho_1 & z \geq H \\ \frac{1}{2}(\rho_1 + \rho_2) & |z| < H \\ \rho_2 & z \leq -H \end{cases} .$$



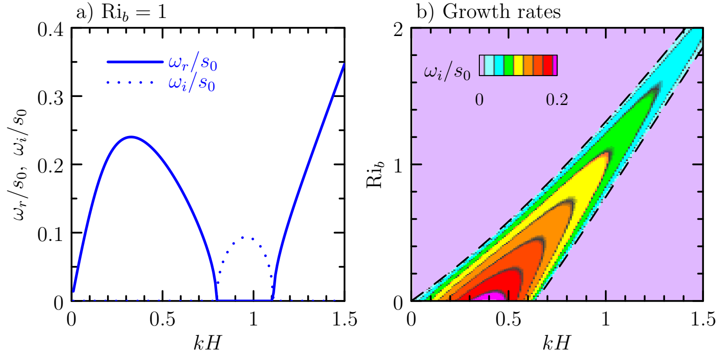
- Proceeding as before, though with considerable more algebra, we find the dispersion relation and diagnose for which k, the frequency ω is complex-valued:

$$\begin{aligned} \tilde{\omega}^4 \quad &- \quad \frac{1}{4}\left[8\tilde{k}^2 - 4(1-\mathrm{Ri}_b)\tilde{k} + 1 - e^{-4\tilde{k}}\right]\tilde{\omega}^2 \\ &+ \quad \frac{1}{4}\tilde{k}^2\left[(2\tilde{k} - 1 - \mathrm{Ri}_b)^2 - (1+\mathrm{Ri}_b)^2 e^{-4\tilde{k}}\right], \end{aligned}$$

in which the frequency and wavenumber have been expressed nondimensionally by $\tilde{\omega} = \omega/s_0$ and $\tilde{k} = kH$.

Kelvin-Helmholtz Instability (cont'd)

- The wavenumber and growth rate of the most unstable mode depends upon the strength of the stratification, as measured by the bulk-Richardson number $\mathrm{Ri}_b \equiv (g'/H)/{s_0}^2$.
- The values become those of unstratified Kelvin-Helmholtz instability as $\mathrm{Ri}_b \to 0$.



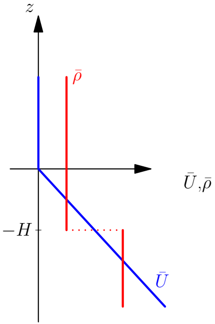
Holmboe Instability

- Now consider a kinked shear in a 2-layer fluid where the density interface does not coincide with the kink in the shear:

$$\bar{\rho} = \begin{cases} \rho_1 & z \geq -H \\ \rho_2 & z < -H \end{cases} .$$

and

$$\bar{U} = \begin{cases} 0 & z \geq 0 \\ -s_0 z & z < 0. \end{cases}$$



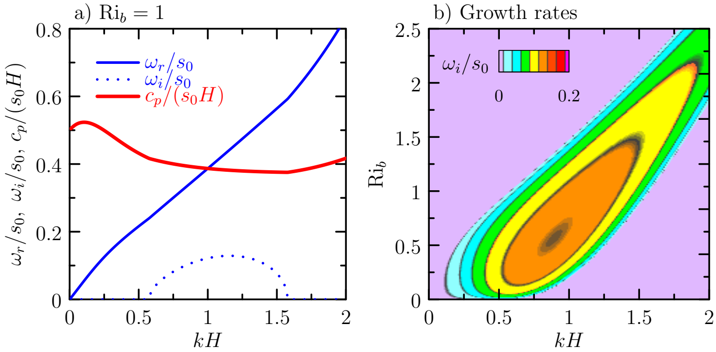
- For this problem, the dispersion relation is

$$\tilde{\omega}^3 - \left(2\tilde{k} + \frac{1}{2}\right)\tilde{\omega}^2 + \tilde{k}\left(\tilde{k} + 1 - \frac{1}{2}\mathrm{Ri}_b\right)\tilde{\omega} - \frac{1}{2}\tilde{k}\left(\tilde{k} - \frac{1}{2}\mathrm{Ri}_b\left[1 - e^{-2\tilde{k}}\right]\right) = 0,$$

in which $\tilde{\omega} \equiv \omega/s_0$ and $\tilde{k} \equiv kH$.

Holmboe Instability (cont'd)

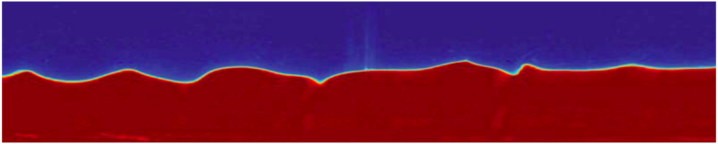
- The flow is unstable only if $\mathrm{Ri}_b > 0$, and the most unstable mode translates rightward.
- The instability arises from a coupling between the interfacial wave centred at $z = -H$ and the shear-Rayleigh wave at $z = 0$.



Holmboe vs Kelvin-Helmholtz

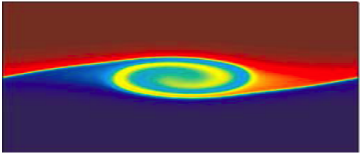
- Holmboe instability is characterized by cusp-shaped waves.
- Whereas Kelvin-Helmholtz instability rapidly mixes, simulations show that Holmboe instability mixes more but on a longer timescale

Experiments showing Holmboe instabilities above and below a shear layer



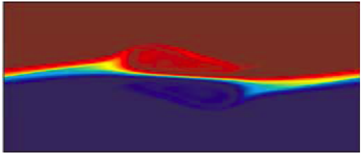
Carpenter et al, J. Fluid Mech. (2010)

Simulation of KH Instability



Carpenter, Lawrence and Smyth, J. Fluid Mech. (2007)

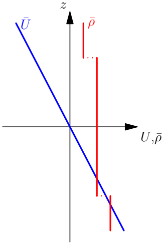
Simulation of Holmboe Instability

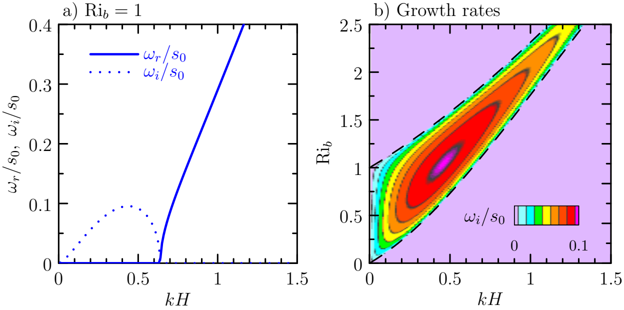


Carpenter, Lawrence and Smyth, J. Fluid Mech. (2007)

Taylor-Caulfield Instability

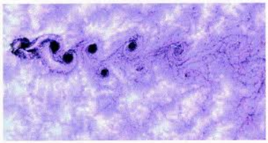
- Now consider a 3-layer fluid in uniform shear.
- As with Holmboe waves, the flow is unstable only if $\text{Ri}_b > 0$.
- Here the instability results from interacting interfacial waves on the upper and lower interfaces, with the instability drawing energy from the background shear.





Other Instabilities

- Spatial Instability
  - solve dispersion relation for (possibly complex) k with given real ω.
  - eg results from shear flows initiated at a fixed location



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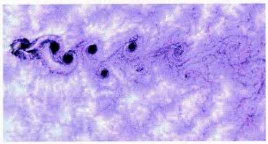
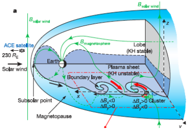
  - solve dispersion relation for (possibly complex) k with given real ω.
  - eg results from shear flows initiated at a fixed location
- Absolute-Convective Instability
  - suppose ω and k are both complex and seek value when $d\omega/dk = 0$.
  - determines if instability is grows in place (absolute) or swept downstream (convective).





Hasegawa et al, Nature (2004)

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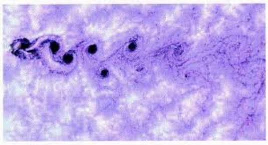

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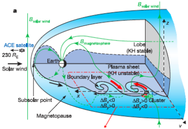
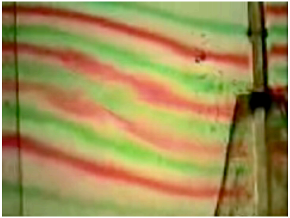


Hasegawa et al, Nature (2004)

- Parametric subharmonic instability
  - if background large-scale flow is oscillatory in time and space, smaller waves can be resonantly excited

    See talks by Tom Peacock.



[Experiment by Ruddick and Hebert (GRL, 2003)]

Review and Next Lecture

- Much can be learned from the consideration of small-amplitude plane waves and quasi-monochromatic wavepackets
  - wave propagation
  - energy and momentum transport
  - growth and structure of instabilities
- Linear theory provides a starting point to understanding realistic systems.
- However, wave breaking with consequent mixing and momentum deposition are inherently nonlinear processes.
  - weak nonlinearity can result in stable structures (eg solitary waves) and unstable structures (eg modulationally unstable waves)
  - fully nonlinear phenomena including wave breaking and wave generation from turbulence lie at the forefront of internal wave research

- Nonlinear considerations will be covered in the next lecture